Tangent lifts of Poisson and related structures

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Abstract. The derivation d_T on the exterior algebra of forms on a manifold M with values in the exterior algebra of forms on the tangent bundle TM is extended to multivector fields. These tangent lifts are studied with application to the theory of Poisson structures, their symplectic foliations, canonical vector fields and Poisson–Lie groups.

Introduction

A derivation d_T on the exterior algebra of forms on a manifold M with values in the exterior algebra of forms of the tangent bundle TM plays essential role in the calculus of variations ([Tu1, Tu2]) and, in particular, in analytical mechanics. The derivation $d_T\omega$ of the symplectic 2-form of a symplectic manifold (M, ω) provides the tangent bundle TM with a symplectic structure. A vector field $X: M \to TM$ is locally Hamiltonian if its image $X(M)$ is a Lagrangian submanifold of $(TM, d_T\omega)$. The concept of a generalized Hamiltonian system can be introduced as a Lagrangian submanifold of $(TM, d_T\omega)$. The infinitesimal dynamics of a relativistic particle is an example of such a system. The derivation d_T has also an aspect of the total Lie derivative in the exterior algebra of forms: $\mathcal{L}_X\mu = X^* d_T\mu$ (theorem 3.2).

In analytical mechanics Poisson structures play a role that is as important as symplectic structures. The phase space is considered to be a manifold possessing a Poisson structure rather than symplectic one. On the other hand, in the theory of systems with symmetries, much attention is paid to the case of Poisson symmetries, i.e. the symmetry group is a Poisson–Lie group. Poisson–Lie groups are of interest also because of their relation to quantum groups.

A Poisson structure is usually given by a bivector field Λ and, in general, not by a two-form and vanishing of the Schouten bracket corresponds to vanishing of the exterior derivative. This shows that, in order to generalize the mentioned ideas and results from the symplectic to the Poisson case, we need to carry over the discussion from forms to multivector fields. The aim of this paper is to extend d_T to the exterior algebra of multivector fields and to establish relations concerning Poisson structures which correspond to the mentioned above relations in symplectic geometry. We would like to emphasize that *our goal is not to extend the general theory of derivations of forms or vector-valued forms*

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(like in [MCS 1–3, By]) to the case of multivector fields, but purely to get an analogue of d_T.

In sections 1 and 2 we concentrate on the definition of d_T on the exterior algebra of forms. The usual definition of d_T as a commutator $[i_T, d]$ does not emphasize the role of the tangent functor and, since it uses the exterior derivative d, cannot be generalized to the case of multivector fields. An r-form μ on M defines a number of vector bundle morphisms

$$\widetilde{\mu}^i : \bigwedge^i TM \to \bigwedge^{r-i} T^*M.$$

We show that $\widetilde{d_T\mu}^i$ can be obtained from the tangent morphism $T\widetilde{\mu}^i$ by natural transformation ([KMS]) of functors, which generalize the well-known natural transformations $\alpha_M : TT^*M \to T^*TM$ and $\kappa_M : TTM \to TTM$. With this fact, the generalization of d_T to the case of multivector fields becomes obvious and it is given in section 2 (theorem 2.2). It appears that d_T is a v_T-derivation of the exterior algebra of multivector fields on M with values in the exterior algebra of multivector fields on TM, where v_T is the vertical lift. In the case of a vector field X on M the resulting field $d_T X$ is the well-known complete lift (see, e.g., [MFL]).

The basic property of d_T is that it commutes with the Schouten bracket (theorem 2.5), what corresponds to the fact that d_T commutes with d on forms. Thus, if Λ is a bivector field representing a Poisson structure on M, then $d_T\Lambda$ defines a Poisson structure on TM. We observe that $d_T\Lambda$ is the tangent Poisson structure discussed in [SdA, Co1]. In the presented approach certain functorial properties of d_T become quite obvious.

In section 3 we show that, as in the case of forms, d_T plays the role of the total Lie derivative. As a consequence, we can describe in section 6 a canonical vector field on a Poisson manifold as a Lagrangian submanifold with respect to the tangent Poisson structure introduced in section 5 (compare with [SdA]). The presentation of the tangent Poisson structure in section 5 is close to that given by Courant in [Co1, Co2, Co3].

The derivation d_T helps to identify vector bundle morphisms $\nu : T^*M \to TM$, which correspond to Poisson structures (theorem 4.4). This identification is complementary to ones expressed in terms of the Jacobi identity and of the Schouten bracket. What is important is that the condition for ν is expressed in terms of objects and morphisms and does not require any additional general operations like exterior derivative and the Schouten bracket. Hence, it is a subject for functorial treatments.

The remaining part of the paper is devoted to the tangent lift of Poisson–Lie structures and to the analysis of the symplectic foliations of tangent Poisson manifolds. We show in section 7 that the tangent group of a Poisson–Lie group (G, Λ) with the tangent Poisson structure is again a Poisson–Lie group. Its Poisson–Lie algebra is the tangent Poisson–Lie algebra of the Poisson–Lie algebra of (G, Λ) (section 8). In section 9 we define the tangent lift of a generalized foliation and in section 10 we prove that the symplectic foliation of the tangent Poisson manifold $(TM, d_T\Lambda)$ is the tangent lift of the symplectic foliation of (M, Λ).

1. Geometric preliminaries

In this section we define morphisms

$$\kappa_M^r : \bigwedge^r TTM \to T\bigwedge^r TM$$

and

$$\varepsilon_M^r : \bigwedge^r T^*TM \to T\bigwedge^r T^*M$$

which generalize the well-known isomorphisms κ_M and $\varepsilon_M = \alpha_M^{-1}$. Functorial properties of these mappings and their duals are discussed.

Let M be a smooth manifold. By $\tau_M: TM \to M$ we denote the tangent fibration and by $\pi_M: T^*M \to M$ the cotangent fibration. For $r = 0, 1, 2, \ldots$, we define exterior product bundles $\bigwedge^r TM$ and $\bigwedge^r T^*M$ with the canonical projections $\tau_M^r: \bigwedge^r TM \to M$ and $\pi_M^r: \bigwedge^r T^*M \to M$ respectively. For $r = 0$ we have $\bigwedge^0 TM \simeq \bigwedge^0 T^*M \simeq M \times \mathbb{R}$.

There is a collection of canonical pairings

$$\langle, \rangle_M^r: \bigwedge\nolimits^r TM \times_M \bigwedge\nolimits^r T^*M \to \mathbb{R}.$$

By applying the tangent functor to these pairings, we obtain tangent pairings

$$\langle, \rangle_{TM}^{\prime r}: \mathsf{T}\bigwedge\nolimits^r TM \times_{TM} \mathsf{T}\bigwedge\nolimits^r T^*M \to \mathsf{T}\mathbb{R} \to \mathsf{T}_0\mathbb{R} = \mathbb{R}.$$

We used the canonical identification of bundles

$$\mathsf{T}\bigwedge\nolimits^r TM \times_{TM} \mathsf{T}\bigwedge\nolimits^r T^*M \simeq \mathsf{T}(\bigwedge\nolimits^r TM \times_M \bigwedge\nolimits^r T^*M).$$

Let (x^i) be a local coordinate system in M. In the bundles $\bigwedge^r TM$, $\bigwedge^r T^*M$, $\mathsf{T}\bigwedge^r TM$ and $\mathsf{T}\bigwedge^r T^*M$ we have adopted coordinate systems

$$(x^i, \dot{x}^{j_1 \ldots j_r}), \quad (x^i, p_{j_1 \ldots j_r}), \quad (x^i, \dot{x}^{j_1 \ldots j_r}, \delta x^k, \delta \dot{x}^{l_1 \ldots l_r}) \text{ and } (x^i, p_{j_1 \ldots j_r}, \dot{x}^k, \dot{p}_{l_1 \ldots l_r})$$

respectively, where $j_1 < j_2 < \cdots < j_r$, etc. In these coordinates the pairings thus introduced read as follows:

$$\langle, \rangle_M^r: ((x^i, \dot{x}^{j_1 \ldots j_r}), (x^i, p_{l_1 \ldots l_r})) \mapsto \sum_{j_1 < j_2 < \cdots < j_r} \dot{x}^{j_1 \ldots j_r} p_{j_1 \ldots j_r}$$

and

$$\langle, \rangle_{TM}^{\prime r}: ((x^i, \dot{x}^{j_1 \ldots j_r}, \delta x^k, \delta \dot{x}^{l_1 \ldots l_r}), (x^i, p_{j_1 \ldots j_r}, \delta x^k, \dot{p}_{l_1 \ldots l_r}))$$

$$\mapsto \sum_{j_1 < \cdots < j_r} (\delta \dot{x}^{j_1 \ldots j_r} \cdot p_{j_1 \ldots j_r} + \dot{x}^{j_1 \ldots j_r} \cdot \dot{p}_{j_1 \ldots j_r}).$$

For each manifold M there is a canonical diffeomorphism (cf [Tu1])

$$\kappa_M: \mathsf{TT}M \to \mathsf{TT}M$$

which is an isomorphism of vector bundles

$$\tau_{TM}: \mathsf{TT}M \to TM \quad \text{and} \quad \mathsf{T}\tau_M: \mathsf{TT}M \to TM.$$

In particular,

$$\tau_{TM} \circ \kappa_M = \mathsf{T}\tau_M \quad \text{and} \quad \mathsf{T}\tau_M \circ \kappa_M = \tau_{TM}.$$

Regarded as a diffeomorphism of $\mathsf{TT}M$, κ_M is involutive: $\kappa_M^2 = Id_{\mathsf{TT}M}$. By α_M we denote the isomorphism

$$\alpha_M: \mathsf{TT}^*M \to \mathsf{T}^*TM$$

of vector bundles

$$\mathsf{T}\pi_M: \mathsf{TT}^*M \to TM \quad \text{and} \quad \pi_{TM}: \mathsf{T}^*TM \to TM$$

dual to κ_M with respect to pairings $\langle, \rangle_{TM}' = \langle, \rangle_{TM}'^1$ and $\langle, \rangle_{TM} = \langle, \rangle_{TM}^1$:

$$\langle v, \alpha_M(w) \rangle_{TM} = \langle \kappa_M(v), w \rangle_{TM}'.$$

In what follows, we denote α_M^{-1} by ε_M. In the above-introduced local coordinates in $\mathsf{TT}M$, TT^*M and the coordinates $(x^i, \dot{x}^j, \pi_k, \dot{\pi}_l)$ adopted from (x^i) in T^*TM, we have

$$(x^i, \dot{x}^j, \delta x^k, \delta \dot{x}^l) \circ \kappa_M = (x^i, \delta x^j, \dot{x}^k, \delta \dot{x}^l)$$

and

$$(x^i, \dot{x}^j, \pi_k, \dot{\pi}_l) \circ \alpha_M = (x^i, \dot{x}^j, \dot{p}_k, p_l).$$

Now, we generalize these morphisms to the multilinear case. Let $\bigwedge^r_M$ be the wedge product mapping

$$\bigwedge^r_M : \times^r_{\tau_M} TM \to \bigwedge^r TM.$$

We apply the tangent functor to this mapping and get

$$T\bigwedge^r_M : T \times^r_{\tau_M} TM \to T\bigwedge^r TM.$$

Since κ_M extends to an isomorphism of vector bundles

$$\times^r \kappa_M : \times^r_{\tau_{TM}} TTM \to \times^r_{T\tau_M} TTM \simeq T \times^r_{\tau_M} TM$$

we get also

$$T\bigwedge^r_M \circ (\times^r \kappa_M) : \times^r_{\tau_{TM}} TTM \to T\bigwedge^r TM.$$

It is easy to verify that this mapping is multilinear and skew-symmetric and, consequently, defines a morphism κ^r_M of vector bundles over TM:

$$\kappa^r_M : \bigwedge^r TTM \to T\bigwedge^r TM. \tag{1.1}$$

In other words

$$\kappa^r_M \circ \bigwedge^r_{TM} = T\bigwedge^r_M \circ \times^r \kappa_M$$

i.e. the following diagram is commutative:

$$\tag{1.2}$$

Of course, $\kappa^1_M = \kappa_M$ and for a simple r-vector $v_1 \wedge \ldots \wedge v_r$ on TM we have

$$\kappa^r_M(v_1 \wedge \ldots \wedge v_r) = T\bigwedge^r_M(\kappa_M(v_1), \ldots, \kappa_M(v^r)).$$

In local coordinates κ^r_M reads as follows:

$$(x^i, \dot{x}^{j_1\ldots j_r}, \delta x^k, \delta \dot{x}^{l_1\ldots l_r}) \circ \kappa^r_M = \left(x_i, \delta x^{j_1\ldots j_r}, \dot{x}^k, \sum_m \delta x^{l_1\ldots l_{m-1}} \wedge \delta \dot{x}^{l_m} \wedge \delta x^{l_{m+1}\ldots l_r} \right)$$

where $(x^i, \dot{x}^j, \delta x^{k_1\ldots k_n} \wedge \delta \dot{x}^{l_{n+1}\ldots l_r})$ are adopted coordinates in $\wedge^r TTM$, with the obvious identification

$$\delta x^{l_1\ldots l_{m-1}} \wedge \delta \dot{x}^{l_m} \wedge \delta x^{l_{m+1}\ldots l_r} = (-1)^{r-m} \delta x^{l_1\ldots l_{m-1} l_{m+1}\ldots l_r} \wedge \delta \dot{x}^{l_m}.$$

Let $\ ^i\bigwedge^{r-i}_M$ be the wedge product

$$^i\bigwedge^{r-i}_M : \bigwedge^i TM \times_M \bigwedge^{r-i} TM \to \bigwedge^r TM.$$

From the diagram (1.2) we easily get that the diagram

$$\bigwedge^i TTM \times_{TM} \bigwedge^{r-i} TTM \xrightarrow{\kappa_M^i \times \kappa_M^{r-i}} T\bigwedge^i TM \times_{TM} T\bigwedge^{r-i} TM$$

$$^i\wedge_{TM}^{r-i}\Big\downarrow \qquad\qquad\qquad\qquad T^i\wedge_M^{r-i}\Big\downarrow \qquad\qquad (1.3)$$

$$\bigwedge^r TTM \xrightarrow{\quad\kappa_M^r\quad} T\bigwedge^r TM$$

is commutative.

In order to define ε_M^r, a counterpart to κ_M^r, we first consider the mapping

$$T\bigwedge_{T^*M}^r \circ (\times^r \varepsilon_M): \times_{\pi_{TM}}^r T^*TM \to T\bigwedge^r T^*TM.$$

Since it is multilinear and skew-symmetric, it defines a mapping

$$\varepsilon_M^r: \bigwedge^r T^*TM \to T\bigwedge^r T^*M$$

and, as in the case of κ_M^r, we have

$$\varepsilon_M^r \circ \bigwedge_{\pi_{TM}}^r = T\bigwedge_{\pi_M}^r \circ \times^r \varepsilon_M$$

i.e. the diagram

$$\times_{\pi_{TM}}^r T^*TM \xrightarrow{\wedge_{TM}^r} \bigwedge^r T^*TM$$

$$\times^r \varepsilon_M\Big\downarrow \qquad\qquad\qquad \Big\downarrow\varepsilon_M^r \qquad\qquad (1.4)$$

$$T\times_{\tau_M}^r T^*M \xrightarrow{T\wedge_M^r} T\bigwedge^r T^*M$$

is commutative.

Let $(x^i, \dot{x}^j, \pi^{k_1\ldots k_n} \wedge \dot{\pi}^{l_{n+1}\ldots l_r})$ be the adopted coordinate system in $\bigwedge^r T^*TM$. We have

$$(x^i, p_{j_1\ldots j_r}, \dot{x}^k, \dot{p}_{l_1\ldots l_r}) \circ \varepsilon_M^r = \left(x^i, \dot{\pi}_{j_1\ldots j_r}, \dot{x}^k, \sum_m \dot{\pi}_{l_1\ldots l_{m-1}} \wedge \pi_{l_m} \wedge \dot{\pi}_{l_{m+1}\ldots l_r}\right).$$

Here we identified $\dot{\pi}_{l_1\ldots l_{m-1}} \wedge \pi_{l_m} \wedge \dot{\pi}_{l_{m+1}\ldots l_r}$ and $(-1)^{m-1}\pi_{l_m} \wedge \dot{\pi}_{l_1\ldots l_{m-1}l_{m+1}\ldots l_r}$.

We have also a commutative diagram

$$\bigwedge^i T^*TM \times_{TM} \bigwedge^{r-i} T^*TM \xrightarrow{\varepsilon_M^i \times \varepsilon_M^{r-i}} T\bigwedge^i T^*M \times_{TM} T\bigwedge^{r-i} T^*M$$

$$^i\wedge_{TM}^{r-i}\Big\downarrow \qquad\qquad\qquad\qquad T^i\wedge_M^{r-i}\Big\uparrow \qquad\qquad (1.5)$$

$$\bigwedge^r T^*TM \xrightarrow{\quad\varepsilon_M^r\quad} T\bigwedge^r T^*M.$$

By $\kappa_M^{r\,\prime}$ and $\varepsilon_M^{r\,\prime}$ we denote the vector bundle morphism

$$\kappa_M^{r\,\prime}: T\bigwedge^r T^*M \to \bigwedge^r T^*TM$$

and

$$\varepsilon_M^{r\,\prime}: T\bigwedge^r TM \to \bigwedge^r TTM$$

dual to κ_M^r and ε_M^r with respect to the pairings $\langle,\rangle_{TM}^{\prime\,r}$ and $\langle,\rangle_{TM}^r$. In particular, we have $\kappa_M^{1\,\prime} = \kappa_M' = \varepsilon_M^{-1} = \alpha_M$ and $\varepsilon_M^{1\,\prime} = \kappa_M$.

Functorial properties

It is known that κ_M and α_M are natural transformations of iterated functors TT, TT^* and $\mathsf{T}^*\mathsf{T}$, i.e. that for every morphism $\varphi: M \to N$ we have

$$\kappa_N \circ \mathsf{TT}\varphi = \mathsf{TT}\varphi \circ \kappa_M$$

$$\alpha_M \circ \mathsf{TT}^*\varphi = \mathsf{T}^*\mathsf{T}\varphi \circ \bar\alpha_N \qquad (1.6)$$

$$\mathsf{TT}^*\varphi \circ \varepsilon_N = \varepsilon_M \circ \mathsf{T}^*\mathsf{T}\varphi.$$

Note that $\mathsf{T}^*\varphi$ is, in general, not a mapping but a relation only, with the domain $\mathsf{T}^*_{\varphi(M)}N$ and codomain $(\ker \mathsf{T}\varphi)^0$. The morphisms

$$\bigwedge{}^r\mathsf{T}\varphi: \bigwedge{}^r\mathsf{T}M \to \bigwedge{}^r\mathsf{T}N$$

and

$$\bigwedge{}^r\mathsf{T}^*\varphi: \bigwedge{}^r\mathsf{T}^*N \to \bigwedge{}^r\mathsf{T}^*M$$

are defined by relations

$$\bigwedge{}^r\mathsf{T}\varphi \circ \bigwedge{}^r_M = \bigwedge{}^r_N \circ \times^r\mathsf{T}\varphi$$

and

$$\bigwedge{}^r\mathsf{T}^*\varphi \circ \bigwedge{}^r_N = \bigwedge{}^r_M \circ \times^r\mathsf{T}^*\varphi.$$

Theorem 1.1. For $\varphi: M \to N$ we have

$$\kappa_N^r \circ \bigwedge{}^r\mathsf{TT}\varphi = \mathsf{T}\bigwedge{}^r\mathsf{T}\varphi \circ \kappa_M^r$$

and

$$\mathsf{T}\bigwedge{}^r\mathsf{T}^*\varphi \circ \varepsilon_N^r = \varepsilon_M^r \circ \bigwedge{}^r\mathsf{T}^*\mathsf{T}\varphi.$$

Proof. From the definition of κ_N^r it follows that

$$\kappa_N^r \circ \bigwedge{}^r\mathsf{TT}\varphi \circ \bigwedge{}^r_{\mathsf{T}M} = \kappa_N^r \circ \bigwedge{}^r_{\mathsf{T}N} \circ \times^r\mathsf{TT}\varphi = \mathsf{T}\bigwedge{}^r_N \circ \times^r\kappa_N \circ \times^r\mathsf{TT}\varphi$$

$$= \mathsf{T}\bigwedge{}^r_N \circ \times^r(\kappa_N \circ \mathsf{TT}\varphi) = \mathsf{T}\bigwedge{}^r_N \circ \times^r(\mathsf{TT}\varphi \circ \kappa_M)$$

$$= \mathsf{T}\bigwedge{}^r_N \circ \times^r\mathsf{TT}\varphi \circ \times^r\kappa_M.$$

Since

$$\bigwedge{}^r\mathsf{T}\varphi \circ \bigwedge{}^r_M = \bigwedge{}^r_N \circ \times^r\mathsf{T}\varphi$$

we get

$$\mathsf{T}\bigwedge{}^r\mathsf{T}\varphi \circ \mathsf{T}\bigwedge{}^r_M = \mathsf{T}\bigwedge{}^r_N \circ \times^r\mathsf{TT}\varphi$$

and

$$\mathsf{T}\bigwedge{}^r_N \circ \times^r\mathsf{TT}\varphi \circ \times^r\kappa_M = \mathsf{T}\bigwedge{}^r\mathsf{T}\varphi \circ \mathsf{T}\bigwedge{}^r_M \circ \times^r\kappa_M = \mathsf{T}\bigwedge{}^r\mathsf{T}\varphi \circ \kappa_M^r \circ \bigwedge{}^r_{\mathsf{T}M}.$$

This completes the proof of the first identity. The proof of the second is analogous. $\square$

We have also the following dual identities.

Theorem 1.2. For $\varphi: M \to N$ we have

$$\kappa_M^{r\,\prime} \circ T\bigwedge{}^r T^* \varphi = \bigwedge{}^r T^* T\varphi \circ \kappa_N^{r\,\prime}$$

and

$$\bigwedge{}^r TT\varphi \circ \varepsilon_M^{r\,\prime} = \varepsilon_N^{r\,\prime} \circ T\bigwedge{}^r T\varphi.$$

2. Derivation d_T of differential forms and multivector fields

In this section we refer to the theory of derivations of differential forms as presented in [PiTu]. We define the derivation d_T on forms in a way which differs from the standard one, but which shows its obvious extension to the case of multivector fields. It appears that defined operation d_T on multivector fields is a derivation of degree zero with respect to the vertical lift of multivector fields. The most important property of d_T is that it commutes with the Schouten bracket.

Definition. Let $\Phi = \bigoplus_{q=0}^{\infty} \Phi^q$ and $\Psi = \bigoplus_{q=0}^{\infty} \Psi^q$ be commutative graded algebras and let $\rho: \Phi \to \Psi$ be a graded algebra homomorphism. A linear mapping $a: \Phi \to \Psi$ is called a ρ-derivation of degree r if

$$a(\Phi^q) \subset \Psi^{q+r}$$

and

$$a(\mu \wedge \nu) = a(\mu) \wedge \rho(\nu) + (-1)^{qr} \rho(\mu) \wedge a(\nu)$$

where $q = \text{degree } \mu$.

Let M be a manifold and let $\tau: E \to M$ be a vector fibration. By $\Phi(\tau)$ we denote the graded exterior algebra generated by sections of τ. For $\tau = \pi_M$ we get the graded algebra of forms on the manifold M and for $\tau = \tau_M$- the graded algebra of multivector fields on M.

Let be $\mu \in \Phi^r(\pi_M)$, i.e. μ is an r-form on M. The *vertical lift* of μ is an r-form $v_T(\mu) \in \Phi^r(\pi_{TM})$. $v_T(\mu) = \tau_M^* \mu$, i.e. $v_T(\mu)$ is the pull-back of μ with respect to the projection τ_M. Since the pull-back commutes with the wedge product, the mapping

$$v_T: \Phi(\pi_M) \to \Phi(\pi_{TM})$$

is a homomorphism of graded commutative algebras.

A *second-order vector field* Γ on M is a vector field on TM such that

$$\tau_{TM} \circ \Gamma = \text{id}_{TM} = (T\tau_M) \circ \Gamma$$

or, equivalently, $\kappa_M \circ \Gamma = \Gamma$. In adopted local coordinates

$$\Gamma(x, \dot{x}) = \dot{x}^k \frac{\partial}{\partial x^k} + f^k(x, \dot{x}) \frac{\partial}{\partial \dot{x}^k}.$$

The contraction of the vertical lift of a form $\mu \in \Phi^r(\pi_M)$, $r > 0$, with Γ does not depend on the choice of the second-order field Γ. We define

$$i_T \mu = \begin{cases} i_\Gamma(v_T\mu) \in \Phi^{r-1}(\pi_{TM}) & \text{for } r > 0 \\ 0 & \text{for } r = 0. \end{cases}$$

The *tangent lift* $d_T\mu$ of $\mu \in \Phi(\pi_M)$ is defined by

$$d_T\mu = d i_T\mu + i_T d\mu = \mathcal{L}_\Gamma v_T(\mu).$$

Since $\mathcal{L}_\Gamma$ is a derivation in $\Phi(\pi_{TM})$ and v_T is a homomorphism of graded algebras, it follows that d_T is a v_T-derivation of degree 0.

If, in local coordinates, $\mu = \mu_{i_1 \ldots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, then

$$d_T \mu(x, \dot{x}) = \frac{\partial \mu_{i_1 \ldots i_r}}{\partial x^k}(x) \dot{x}^k dx^{i_1} \wedge \ldots \wedge dx^{i_r} + \sum_m \mu_{i_1 \ldots i_r}(x) dx^{i_1} \wedge \ldots \wedge d\dot{x}^{i_m} \wedge \ldots dx^{i_r}$$

(2.1)

for $r > 0$ and

$$d_T \mu(x, \dot{x}) = \frac{\partial \mu}{\partial x^i}(x) \dot{x}^i$$

for $r = 0$.

The operation i_T, which is, in fact, a v_T-derivation of degree -1, can be defined in a more intrinsic way. An r-form μ, $r > 0$, defines a vector bundle morphism

$$\tilde{\mu}^1 : TM \to \bigwedge^{r-1} T^*M : v \mapsto i_v \mu$$

and the following formula holds:

$$i_T \mu = (\tilde{\mu}^1)^* \theta_M^{r-1}$$

(2.2)

where θ_M^{r-1} is the canonical (Liouville) $(r-1)$-form on $\bigwedge^{r-1} T^*M$. The Liouville form is defined by

$$\theta_M^{r-1}(a)(v_1, \ldots, v_{r-1}) = a(T\pi_M(v_1), \ldots T\pi_M(v_{r-1})).$$

Let us notice also that for $r = 0$ we have $d_T \mu(v) = \langle v, d\mu \rangle$ (μ is a function).

The tangent lift $d_T \mu$ can be defined more directly by means of the tangent functor. Let us fix $0 \leqslant i \leqslant r$. An r-form μ on M defines, in an obvious way, a vector bundle morphism

$$\tilde{\mu}^i : \bigwedge^i TM \to \bigwedge^{r-i} T^*M.$$

Now, we define κ_M^r and ε_M^r for $r = 0$. We have $\bigwedge^0 TM = \bigwedge^0 T^*M = M \times \mathbb{R}$. We define

$$\kappa_M^0 = \varepsilon_M^0 : \bigwedge^0 TTM = TM \times \mathbb{R} \to T\bigwedge^0 TM = T(M \times \mathbb{R}) = TM \times \mathbb{R} \times T_0\mathbb{R}$$

by

$$(x^i, \dot{x}^j, t, \dot{t}) \circ \kappa_M^0 = (x^i, \dot{x}^j, 0, t).$$

The dual mapping $\kappa_M^{0\,\prime} : T\bigwedge^0 TM \to \bigwedge^0 TTM$ is given by

$$(x^i, \dot{x}^j, t) \circ \kappa_M^{0\,\prime} = (x^i, \dot{x}^j, t).$$

Theorem 2.1. Let be $\mu \in \Phi^r(\pi_M)$ and $0 \leqslant i \leqslant r$. The following diagram is commutative:

$$
\begin{array}{ccc}
T\bigwedge^i TM & \xrightarrow{\;T\tilde{\mu}^i\;} & T\bigwedge^{r-i} T^*M \\[1em]
\kappa_M^i \uparrow & & (\kappa_M^{r-i})' \downarrow \\[1em]
\bigwedge^i TTM & \xrightarrow{\;\widetilde{d_T\mu}^i\;} & \bigwedge^{r-i} T^*TM \,.
\end{array}
$$

(2.3)

Proof. We show first that there exists $D(\mu) \in \Phi^r(\pi_{TM})$ such that

$$\widetilde{D(\mu)}^i = (\kappa_M^{r-i})' \circ T\widetilde{\mu}^i \circ \kappa_M^i.$$

In order to do this, we apply the tangent functor to the commutative diagram

$$
\begin{array}{ccc}
\bigwedge^r TM & \xrightarrow{\quad \widetilde{\mu}^r \quad} & \mathbb{R} \\
{\scriptstyle i}\bigwedge_M^{r-i}\uparrow & & \langle,\rangle_M^{r-i}\uparrow \\
\bigwedge^i TM \times_M \bigwedge^{r-i} TM & \xrightarrow{\widetilde{\mu}^i \times \mathrm{id}} & \bigwedge^{r-i} T^*M \times_M \bigwedge^{r-i} TM
\end{array}
$$
(2.4)

and, since the diagram (1.5) is commutative, we get the following commutative diagram:

$$
\begin{array}{ccccc}
\bigwedge^r TTM & \xrightarrow{\kappa_M^r} & T\bigwedge^r TM & \xrightarrow{d_T\widetilde{\mu}^r} & \mathbb{R} \\
{\scriptstyle i}\bigwedge_{TM}^{r-i}\uparrow & & T^i\bigwedge_M^{r-i}\uparrow & & \langle,\rangle_{TM}^{r-i}\uparrow \\
\bigwedge^i TTM \times_{TM} \bigwedge^{r-i} TTM & \xrightarrow{\kappa_M^i \times \kappa_M^{r-i}} & T\bigwedge^i TM \times_{TM} T\bigwedge^{r-i} TM & \xrightarrow{T\widetilde{\mu}^i \times \mathrm{id}} & T\bigwedge^{r-i} T^*M \times_{TM} T\bigwedge^{r-i} TM .
\end{array}
$$

(2.5)

Here we regard 0-forms as functions rather than sections of $\bigwedge^0 T$–bundles. It shows that for $u \in \bigwedge^i TTM$ and $v \in \bigwedge^{r-i} TTM$ we have

$$\langle T\widetilde{\mu}^i \circ \kappa_M^i(u), \kappa_M^{r-i}(v)\rangle_{TM}^{r-i} = d_T\widetilde{\mu}^r \circ \kappa_M^r(u \wedge v)$$

and, consequently,

$$d_T\widetilde{\mu}^r \circ \kappa_M^r(u \wedge v) = \langle(\kappa_M^{r-i})' \circ T\widetilde{\mu}^i \circ \kappa_M^i(u), v\rangle_{TM}^{r-i}.$$

It follows that $D(\mu)$ exists and $D(\mu)(\cdot) = d_T\widetilde{\mu}^r \circ \kappa_M^r$. Since in local coordinates

$$\widetilde{\mu}^r(x, \dot{x}) = \mu_{i_1\ldots i_r}\dot{x}^{i_1\ldots i_r}$$

we get

$$d_T\widetilde{\mu}^r(x, \dot{x}, \delta x, \delta\dot{x}) = \frac{\partial\mu_{i_1\ldots i_r}}{\partial x^k}(x)\delta x^k\dot{x}^{i_1\ldots i_r} + \mu_{i_1\ldots i_r}(x)\delta\dot{x}^{i_1\ldots i_r}$$

and

$$d_T\widetilde{\mu}^r \circ \kappa_M^r = \frac{\partial\mu_{i_1\ldots i_r}}{\partial x^k}(x)\dot{x}^k\delta x^{i_1\ldots i_r} + \sum_m \mu_{i_1\ldots i_r}(x)\delta x^{i_1\ldots i_{m-1}} \wedge \delta\dot{x}^{i_m} \wedge \delta x^{i_{m+1}\ldots i_r}.$$

The right-hand side of this formula corresponds precisely to the right-hand side of (2.1). It follows that $d_T\widetilde{\mu}^r \circ \kappa_M^r = \widetilde{d_T\mu}^r$ and this completes the proof. $\square$

Derivations of multivector fields and the Schouten bracket

A similar construction can be done in the case of multivector fields. Let be $X \in \Phi^r(\tau_M)$. As in the case of forms, we have a family of contraction mappings

$$\widetilde{X}^i: \bigwedge^i T^*M \to \bigwedge^{r-i} TM.$$

Theorem 2.2.   There is a uniquely defined multivector field $d_T X \in \Phi^r(\tau_{TM})$ such that

$$\widetilde{d_T X}^i = (\varepsilon_M^{r-i})' \circ T\widetilde{X}^i \circ \varepsilon_M^i$$

for $i = 0, 1, \ldots, r$, i.e. the following diagram is commutative

$$
\begin{array}{ccc}
T\bigwedge^i T^* M & \xrightarrow{\ T\widetilde{X}^i\ } & T\bigwedge^{r-i} TM \\[2mm]
\varepsilon_M^i \big\uparrow & & \big\downarrow (\varepsilon_M^{r-i})' \\[2mm]
\bigwedge^i T^* TM & \xrightarrow{\ \widetilde{d_T X}^i\ } & \bigwedge^{r-i} TTM.
\end{array}
\qquad (2.6)
$$

The proof follows that of theorem 2.1. In particular, we get

$$\widetilde{d_T X}^r = d_T \widetilde{X}^r \circ \varepsilon_M^r$$

i.e.

$$d_T X(a_1, \ldots, a_r) = d_T \widetilde{X}^r(\varepsilon_M a_1 \wedge \ldots \wedge \varepsilon_M a_r).$$

Now, writing in local coordinates

$$X = X^{i_1 \ldots i_r}(x) \frac{\partial}{\partial x_{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x_{i_r}}$$

and $p = p_{i_1 \ldots i_r} \, dx^{i_1} \wedge \cdots \wedge dx^{i_r}$, we have

$$\widetilde{X}^r(p) = X^{i_1 \ldots i_r} p_{i_1 \ldots i_r}$$

and

$$d_T \widetilde{X}^r(x^i, \dot{x}^k, \pi_{i_1 \ldots i_m} \wedge \dot{\pi}_{i_{m+1} \ldots i_r})$$

$$= \frac{\partial X^{i_1 \ldots i_r}}{\partial x^{i_k}}(x) \dot{x}^k \pi_{i_1 \ldots i_r} + \sum_m X^{i_1 \ldots i_r}(x) \dot{\pi}_{i_1 \ldots i_{m-1}} \wedge \pi_{i_m} \wedge \dot{\pi}_{i_{m+1} \ldots i_r}$$

i.e.

$$d_T X = \frac{\partial X^{i_1 \ldots i_r}}{\partial x^{i_k}}(x) \dot{x}^k \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial \dot{x}^{i_r}} + \sum_m X^{i_1 \ldots i_r}(x) \frac{\partial}{\partial \dot{x}^{i_1}} \wedge \cdots \wedge \frac{\partial}{\partial x^{i_m}} \wedge \cdots \wedge \frac{\partial}{\partial \dot{x}^{i_r}}.$$

Now, let X be a simple r-vector field, i.e. $X = X_1 \wedge \cdots \wedge X_r$ for some vector fields $X_i \in \Phi(\tau_M)$. We can consider $\widetilde{X}^r$ as a multilinear, skewsymmetric function on $\times_M^r T^* M$

$$\widetilde{X}^r(a_1, \ldots, a_r) = \sum_{\sigma \in S_n} (-1)^\sigma \widetilde{X_{\sigma(1)}}^1(a_1) \cdots \widetilde{X_{\sigma(r)}}^1(a_r).$$

Since d_T is a $v_T = \tau^*_{\times^r T^* M}$-derivation on forms, we have

$$d_T \widetilde{X}^r = \sum_{\sigma \in S_n} (-1)^\sigma \sum_m v_T(\widetilde{X_{\sigma(1)}}^1) \cdots d_T(\widetilde{X_{\sigma(m)}}^1) \cdots v_T(\widetilde{X_{\sigma(r)}}^1)$$

and

$$\widetilde{d_T X}^r = \sum_{\sigma \in S_n} (-1)^\sigma \sum_m (v_T(\widetilde{X_{\sigma(1)}}^1)) \circ \varepsilon_M \cdots (d_T(\widetilde{X_{\sigma(m)}}^1)) \circ \varepsilon_M \cdots (v_T(\widetilde{X_{\sigma(r)}}^1)) \circ \varepsilon_M.$$

Let $Y \in \Phi^1(\tau_M)$ be a vector field on M. The functions $v_T(\widetilde{Y}^1) \circ \varepsilon_M$ and $d_T(\widetilde{Y}^1) \circ \varepsilon_M$ are linear functions on T^*TM and define vector fields on TM. In local coordinates we have the following for $Y = Y^i \partial/\partial x_i$:

$$\widetilde{Y}^1(x, p) = Y^i(x)p_i \qquad \text{and} \qquad v_T(\widetilde{Y}^1)(x, p, \dot{x}, \dot{p}) = Y^i(x)p_i.$$

Hence, from the definition of ε_M,

$$v_T(\widetilde{Y}^1) \circ \varepsilon_M(x, \dot{x}, \pi, \dot{\pi}) = Y^i(x)\dot{\pi}_i.$$

It follows that $v_T(\widetilde{Y}^1) \circ \varepsilon_M = \widetilde{Y^v}^1$, where $Y^v = Y^i(x)\partial/\partial \dot{x}^i$ is the *vertical lift* of Y. The vertical lift $\Phi^1(\tau_M) \ni Y \mapsto Y^v \in \Phi^1(\tau_{TM})$ is linear and can be extended in a unique way to a homomorphism v_T of graded algebras

$$v_T: \Phi(\tau_M) \rightarrow \Phi(\tau_{TM}).$$

The vertical lift Y^v is the generator of a one-parameter group (ψ^t) of diffeomorphisms (a flow) of TM defined by

$$\psi^t(v) = v + tY(\tau_M(v)).$$

In a similar way we get

$$d_T\widetilde{Y}^1(x, p, \dot{x}, \dot{p}) = Y^i(x)\dot{p}_i + \frac{\partial Y^i}{\partial x_k}(x)p_i\dot{x}^k$$

and

$$d_T\widetilde{Y}^1 \circ \varepsilon_M(x, \dot{x}, \pi, \dot{\pi}) = Y^i(x)\pi_i + \frac{\partial Y^i}{\partial x^k}(x)\dot{x}^k\dot{\pi}_i.$$

It follows that $d_T\widetilde{Y}^1 \circ \varepsilon_M = \widetilde{Y^c}^1$, where

$$Y^c(x, \dot{x}) = Y^i(x)\frac{\partial}{\partial x^i} + \frac{\partial Y^i}{\partial x^k}(x)\dot{x}^k\frac{\partial}{\partial \dot{x}^i}$$

is the *complete lift* of Y. The vector field Y^c is the generator of the (local) one-parameter group of diffeomorphisms $T\varphi^t: TM \rightarrow TM$, where (φ^t) is the flow generated by the vector field Y.

Thus we have proved the following theorem.

Theorem 2.3. The mapping

$$d_T: \Phi(\tau_M) \rightarrow \Phi(\tau_{TM})$$

is a v_T-derivation of degree 0 with $d_T(Y) = Y^c$ for $Y \in \Phi^1(\tau_M)$.

There follows a well-known theorem (e.g. [MFL, Co1, Co2]).

Theorem 2.4. The Lie bracket of vertical and complete lifts satisfy the following commutation relations

$$[X^v, Y^v] = 0$$
$$[X^c, Y^c] = [X, Y]^c \qquad\qquad (2.7)$$
$$[X^v, Y^c] = [X, Y]^v.$$

The Lie bracket of vector fields can be extended to a graded Lie bracket on the graded space $\Phi(\tau_M)$ of multivector fields—the Schouten bracket $[\cdot, \cdot]$. In this graded Lie algebra the space $\Phi^r(\tau_M)$ is of degree $(r-1)$. Let be $X \in \Phi^r(\tau_M)$. Then

$$[X, Y \wedge Z] = [X, Y] \wedge Z + (-1)^{s(r-1)} Y \wedge [X, Z]$$

for $Y \in \Phi^s(\tau_M)$, i.e. ad_X is a graded derivation of degree $(r-1)$ of the graded commutative algebra $\Phi(\tau_M)$. The mapping

$$\mathrm{ad} : (\Phi(\tau_M), [\cdot, \cdot]) \to \mathrm{Der}\,(\Phi(\tau_M), \wedge)$$

is a homomorphism of graded algebras. For simple multivectors we have the following formula (cf [Mi]):

$$[X_1 \wedge \cdots \wedge X_p, \; Y_1 \wedge \cdots \wedge Y_q]$$

$$= \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \cdots X_{i-1} \wedge X_{i+1} \wedge \cdots \wedge Y_{j-1} \wedge Y_{j+1} \wedge \cdots Y_q.$$

$$(2.8)$$

Theorem 2.5.   The derivation d_T commutes with the Schouten bracket, i.e.

$$[d_T X, d_T Y] = d_T [X, Y].$$

Proof.   Let be $X = X_1 \wedge \cdots \wedge X_p$. By $(X_1, \ldots, X_p)_n$ we denote a p-vector field $X_1^v \wedge \cdots \wedge X_n^c \wedge \cdots \wedge X_p^v$ and by $(X_1, \ldots, X_p)_n^i$ a $(p-1)$-vector field as above with the ith factor omitted. We then have $d_T X = \sum_{i=1}^p (X_1, \ldots, X_p)_i$. From the formula (2.8) and theorem 2.3 we get, for $Y = Y_1 \wedge \ldots \wedge Y_q$,

$$[d_T X, d_T Y] = \left[\sum_{n=1}^p (X_1, \ldots, X_p)_n, \; \sum_{m=1}^q (Y_1, \ldots, Y_p)_m \right]$$

$$= \sum_{n=1}^p \sum_{m=1}^q [(X_1, \ldots, X_p)_n, (Y_1, \ldots, Y_p)_m]$$

$$= \sum_{n=1}^p \sum_{m=1}^q \left(\sum_{j \neq m} (-1)^{n+j} [X_n^c, Y_j^v] \wedge (X_1, \ldots, X_p)_n^n \wedge (Y_1, \ldots, Y_q)_m^j \right.$$

$$+ (-1)^{n+m} [X_n^c, Y_m^c] \wedge (X_1, \ldots, X_p)_n^n \wedge (Y_1, \ldots, Y_p)_m^m$$

$$+ \left. \sum_{i \neq m} (-1)^{m+i} [X_i^v, Y_m^c] \wedge (X_1, \ldots, X_p)_n^i \wedge (Y_1, \ldots, Y_q)_m^m \right)$$

$$= \sum_{n,j} (-1)^{n+j} [X_n, Y_j]^v \wedge \sum_{m \neq j} (X_1, \ldots, X_p)_n^n \wedge (Y_1, \ldots, Y_q)_m^j$$

$$+ (-1)^{n+m} [X_n, Y_m]^c \wedge (X_1, \ldots, X_p)_n^n \wedge (Y_1, \ldots, Y_p)_m^m$$

$$+ \sum_{m,i} (-1)^{i+m} [X_i, Y_m]^v \wedge \sum_{n \neq i} (X_1, \ldots, X_p)_n^i \wedge (Y_1, \ldots, Y_q)_m^m.$$

On the other hand,

$$d_T[X, Y] = \sum_{i,j}(-1)^{i+j}[X_i, Y_j]^c \wedge (X_1, \ldots, X_p)_i^i \wedge (Y_1, \ldots, Y_q)_j^j$$

$$+ \sum_{i,j}(-1)^{i+j}[X_i, Y_j]^v \wedge \sum_{m \neq i}(X_1, \ldots, X_p)_m^i \wedge (Y_1, \ldots, Y_q)_j^j$$

$$+ \sum_{i,j}(-1)^{i+j}[X_i, Y_j]^v \wedge \sum_{n \neq j}(X_1, \ldots, X_p)_i^i \wedge (Y_1, \ldots, Y_q)_n^j$$

and the required equality follows.   □

We list the rules of contractions for lifts of forms and vector fields in the following proposition.

Proposition 2.6. For $\mu \in \Phi^1(\pi_M)$ and $X \in \Phi^1(\tau_M)$ we have

$$\langle v_T(\mu), v_T(X) \rangle = \langle \tau_M^* \mu, X^v \rangle = 0$$

$$\langle d_T\mu, v_T(X) \rangle = \langle v_T\mu, d_T(X) \rangle = v_T(\langle \mu, X \rangle) = \tau_M^*\langle \mu, X \rangle$$

$$\langle d_T\mu, d_TX \rangle = d_T(\langle \mu, X \rangle).$$

These formulae can easily be generalized for $X \in \Phi^r(\tau_M)$ (or $\mu \in \Phi^r(\pi_M)$). We obtain

$$\langle d_TX, v_T\mu \rangle \stackrel{(rmdef)}{=} i_{v_T(\mu)}d_TX = v_T(i_\mu X)$$

$$\langle d_TX, d_T\mu \rangle \stackrel{(def)}{=} i_{d_T(\mu)}d_TX = d_T(i_\mu X) \qquad\qquad (2.9)$$

$$\langle v_T(X), v_T(\mu) \rangle \stackrel{(def)}{=} i_{v_T(\mu)}v_TX = 0$$

for $X \in \Phi^r(\tau_M)$, $\mu \in \Phi^1(\pi_M)$ and also

$$\langle d_T\mu, d_TX \rangle = 0 \qquad \text{for} \quad \mu \in \Phi^r(\pi_M) \quad X \in \Phi^r(\tau_M) \quad r > 2.$$

3. Lie derivations of forms and multivector fields

The derivation d_T on forms is strictly related to the Lie derivation in the algebra of differential forms. It can be considered as a total Lie derivative. This point of view is justified by proposition 3.1. Theorem 3.2 gives an analogous result for d_T on multivector fields.

Proposition 3.1. ([PiTu]). Let $X: M \to TM$ be a vector field on M. Then we have

$$\pounds_X\mu = X^*d_T\mu. \qquad\qquad (3.1)$$

Proof. Since $X^*: \Phi(\pi_{TM}) \to \Phi(\pi_M)$ is a homomorphism and $d_T: \Phi(\pi_M) \to \Phi(\pi_{TM})$ is a v_T-derivation, the mapping

$$X^* d_T: \Phi(\pi_M) \to \Phi(\pi_M) \qquad (3.2)$$

is also a derivation. It follows that it is enough to verify the formula for functions and their differentials. We have for a function f on M

$$X^* d_T f = d_T f \circ X = \langle X, df \rangle = X(f)$$

and

$$X^* d_T df = X^* dd_T f = d(X^* d_T f) = d(X(f)) = \pounds_X df.$$

$\square$

To get an analogue to this proposition in the case of vector fields, we need an analogue of the pull-back of forms with respect to a vector field. Let X be a vector field on M. We have the decomposition of $T_{X(M)} TM$ into horizontal (tangent to $X(M)$) and vertical parts

$$T_{X(M)} TM = H_X TM \oplus V_X TM. \qquad (3.3)$$

The canonical projection $T_X TM \to V_X TM$ we denote by P_X. Let $Y \in \Phi^1(\tau_{TM})$ be a vector field on TM. By X^*Y we denote the unique vector field on M such that

$$v_T(X^*Y)|_{X(M)} = P_X Y. \qquad (3.4)$$

The mapping $Y \mapsto X^*Y$ has the unique extension to a morphism of graded algebras

$$X^*: \Phi(\tau_{TM}) \to \Phi(\tau_M). \qquad (3.5)$$

Theorem 3.2. Let $X: M \to TM$ be a vector field on M. For each $\lambda \in \Phi(\tau_M)$ we have the following formulae:

$$\pounds_X \lambda = X^* d_T \lambda$$

and

$$\lambda = X^* v_T \lambda.$$

Proof. Since $X^*: \Phi(\tau_{TM}) \to \Phi(\tau_M)$ is a homomorphism and $d_T: \Phi(\tau_M) \to \Phi(\tau_{TM})$ is a v_T-derivation the mapping

$$X^* d_T: \Phi(\tau_M) \to \Phi(\tau_M) \qquad (3.6)$$

is also a derivation. It follows that it is enough to verify the formula for functions and vector fields. We have for a function f on M and a vector field Y on M

$$X^* d_T f = d_T f \circ X = \langle X, df \rangle = X(f) = \pounds_X f$$

and (proposition 2.6)

$$\tau_M^*(X^* d_T Y(f)) = v_T((X^* d_T Y)(d_T f)) = v_T(X^* d_T Y)(d_T f).$$

On the other hand,

$$(P_X d_T Y) \circ X = d_T Y \circ X - TX \circ Y$$

and, consequently,

$$(X^* d_T Y)(f) = \langle d_T Y, d_T df \rangle \circ X - \langle TX \circ Y, d_T df \rangle.$$

Since

$$\langle d_T Y, d_T df \rangle \circ X = (d_T(\langle Y, df \rangle)) \circ X = X(Y(f))$$

and

$$\langle TX \circ Y, d_T df \rangle = \langle Y, X^* dd_T f \rangle = \langle Y, dX(f) \rangle = Y(X(f))$$

we get

$$(X^* d_T Y)(f) = X(Y(f)) - Y(X(f)) = [X, Y](f) = (\pounds_X Y)(f).$$

The second identity follows directly from the definition. $\qquad\square$

Example

Let be $M = \mathbb{R}$, $X = X(x)\partial/\partial x$ and $Y = Y(x)\partial/\partial x$. We then have

$$d_T Y = Y(x)\frac{\partial}{\partial x} + Y'(x)\dot{x}\frac{\partial}{\partial \dot{x}}$$

and

$$d_T Y|_X = Y(x)\frac{\partial}{\partial x} + Y'(x)X(x)\frac{\partial}{\partial \dot{x}}.$$

The subspace of horizontal vectors is spanned by $\partial/\partial x + X'(x)\partial/\partial \dot{x}$. Hence the decomposition of $d_T Y|_n X$ into horizontal and vertical parts is the following:

$$Y(x)\frac{\partial}{\partial x} + Y'(x)X(x)\frac{\partial}{\partial \dot{x}}$$

$$= Y(x)\left(\frac{\partial}{\partial x} + X'(x)\frac{\partial}{\partial \dot{x}}\right) + (X(x)Y'(x) - Y(x)X'(x))\frac{\partial}{\partial \dot{x}}.$$

The vertical part is obviously the vertical lift of

$$(X(x)Y'(x) - Y(x)X'(x))\frac{\partial}{\partial x} = [X, Y](x) = \pounds_X Y(x).$$

In the following sections we represent a Poisson structure by a vector bundle morphism $T^*M \to TM$ rather than by a bi-vector field. We then need a formula for the Lie derivative of a bi-vector field λ expressed in terms of $\widetilde{\lambda}^1$. In order to get it, we first identify an operation dual to X^*.

The projection $T\tau_M$ induces an isomorphism $T\tau_M : H_X TM \to TM$ and the dual isomorphism $(T\tau_M)' : T^*M \to H_X^* TM$.

When composed with $(I - P_X)'$, this isomorphism gives an isomorphism $T^*M \to V_X^\circ TM$, where $V_X^\circ TM$ is the annihilator of $V_X T_M$ in $T_{X(M)}^* TM$. It follows that for $v \in \Phi^1(\pi_{TM})$ the decomposition

$$v = (v - v_T(X^* v)) + v_T(X^* v)$$

is, on $X(M)$, the decomposition related to the decomposition

$$T_{X(M)}^* TM = H_X^\circ TM \oplus V_X^\circ TM$$

dual to (3.3). For $v = d_T \mu$ we have $v_T(X^* v) = v_T(\pounds_X \mu)$ (proposition 3.1). Hence, for $Y \in \Phi^1(\tau_{TM})$ we have

$$\langle \mu, X^* Y \rangle = (\langle d_T \mu, v_T(X^* Y) \rangle) \circ X = (\langle d_T \mu - v_T(\pounds_X \mu), Y \rangle) \circ X \qquad (3.7)$$

It is convenient to introduce an operation

$$X_+ : \Phi(\pi_M) \to \Phi(\pi_{TM}) : \mu \longmapsto d_T\mu - v_T(\pounds_X\mu)$$

which is a v_T-derivation of degree 0.

It follows directly from (3.7) that for $Y \in \Phi^r(\tau_{TM})$ and $\mu_1, \ldots, \mu_r \in \Phi^1(\pi_M)$, $r > 0$, we have

$$\langle (X_+\mu_1) \wedge \ldots \wedge (X_+\mu_r), Y \rangle \circ X = \langle \mu_1 \wedge \ldots \wedge \mu_r, X^*Y \rangle \qquad (3.8)$$

so the mapping

$$X_*(\mu_1 \wedge \ldots \mu_r) = (X_+\mu_1) \wedge \ldots \wedge (X_+\mu_r)$$

regarded as a homomorhism of graded algebras

$$X_* : \Phi(\pi_M) \to \Phi(\pi_{X(M)})$$

where

$$\pi_{X(M)} : \mathsf{T}^*_{X(M)}\mathsf{T}M \to X(M)$$

is dual to X^* :

$$\langle X_*\mu, Y \rangle \circ X = \langle \mu, X^*Y \rangle.$$

For 1-forms we shall write $X_*\mu$ instead of $X_+\mu$.

Proposition 3.3. Let X be a vector field on M and let λ be a 2-vector field on M. $\pounds_X\lambda = 0$ if and only if for every pair f, g of functions on M

$$\langle X_*dg, \widetilde{d_T\lambda}^1(X_*df) \rangle = 0.$$

Proof. From theorem 3.2 it follows that $\pounds_X\lambda = 0$ iff

$$\langle dg, \widetilde{X^*d_T\lambda}^1(df) \rangle = 0$$

for each pair (f, g) of functions on M. From (3.8) we then have that $\pounds_X\lambda = 0$ if and only if

$$\langle X_*dg, \widetilde{d_T\lambda}^1 X_*df \rangle = 0.$$

$\square$

4. Symplectic and Poisson structures

In this section we give definitions of symplectic and Poisson structures represented by morphisms of tangent and cotangent bundles. These definitions do not make use of the exterior derivative and of the Schouten bracket.

By the standard definition, a symplectic structure on M is a non-degenerate, closed two-form ω on M. The canonical example is the 2-form Ω_M on T^*M. On the other hand, any 2-form μ on M can be represented by

$$\tilde{\mu}^i : \bigwedge^i \mathsf{T}M \to \bigwedge^{2-i} \mathsf{T}^*M \qquad i = 0, 1, 2.$$

The standard definition is expressed, in fact, in terms of $\tilde{\mu}^0$. We can also formulate a definition of a symplectic structure in terms of $\tilde{\mu}^2$ making use of the well known formula for the exterior derivative:

Proposition 4.1. A linear fibre bundle morphism $\nu: \bigwedge^2 TM \to \bigwedge^0 T^*M$ represents a symplectic structure on M if it is non-degenerate (in the obvious meaning) and

$$X(\nu(Y, Z)) + Y(\nu(Z, X)) + Z(\nu(X, Y)) - \nu([X, Y], Z) - \nu([Y, Z], X)$$

$$-\nu([Z, X], Y) = 0$$

where $X, Y, Z \in \Phi^1(\tau_M)$.

We now provide a definition of a (pre-)symplectic structure in terms of $\widetilde{\mu}^1$. We first need a condition for a 2-form μ to be closed in terms of $\widetilde{\mu}^1$.

Theorem 4.2. A 2-form μ on M is closed if and only if

$$d_T \mu = (\widetilde{\mu}^1)^* \Omega_M$$

Proof. From the formula 2.2 we have

$$d i_T \mu = (\widetilde{\mu}^1)^* \Omega_M$$

and, consequently,

$$d_T \mu = d i_T \mu + i_T d\mu = (\widetilde{\mu}^1)^* \Omega_M - i_T d\mu.$$

Since $i_T d\mu = 0$ iff $d\mu = 0$, we get that $d\mu = 0$ if and only if

$$d_T \mu = (\widetilde{\mu}^1)^* \Omega_M.$$

$\square$

A condition for a vector bundle morphism $\nu: TM \to T^*M$ to define a symplectic structure on M is given by the following theorem.

Theorem 4.3. An isomorphism $\nu: TM \to T^*M$ of vector bundles defines a symplectic structure on M if and only if the following diagram is commutative:

$$
\begin{array}{ccc}
\mathsf{TT}^*M & \xrightarrow{\widetilde{\Omega_M}^1} & \mathsf{T}^*\mathsf{T}^*M \\[4pt]
{\scriptstyle T\nu}\big\uparrow & & {\scriptstyle T^*\nu}\big\downarrow \\[4pt]
\mathsf{TT}M & & \mathsf{T}^*\mathsf{T}M \\[4pt]
{\scriptstyle \kappa_M}\big\downarrow & & {\scriptstyle \alpha_M}\big\uparrow \\[4pt]
\mathsf{TT}M & \xrightarrow{\;\;T\nu\;\;} & \mathsf{TT}^*M
\end{array}
\qquad (4.1)
$$

Proof. Commutativity of the diagram (4.1) is equivalent to the equality

$$\alpha_M \circ \mathsf{T}\nu \circ \kappa_M = \mathsf{T}^*\nu \circ \widetilde{\Omega_M}^{-1} \circ \mathsf{T}\nu \tag{4.2}$$

and to the dual (with respect to proper pairings) equality

$$\alpha_M \circ \mathsf{T}\nu^* \circ \kappa_M = \mathsf{T}^*\nu \circ (\widetilde{\Omega_M}^{-1})^* \circ \mathsf{T}\nu. \tag{4.3}$$

Since $(\widetilde{\Omega_M}^{-1})^* = -\widetilde{\Omega_M}$, we get from (4.3) that

$$\mathsf{T}\nu = -\mathsf{T}\nu^*$$

and, consequently, that $\nu^* = -\nu$. It proves that ν is skew-symmetric and, consequently, that the exist a 2-form μ on M, such that $\nu = \widetilde{\mu}^1$. The equality (4.2) now reads

$$d\mu = (\widetilde{\mu}^1)^* \Omega_M.$$

It follows from theorem 4.2 that μ is closed. $\qquad\square$

As in the case of symplectic structures, a Poisson structure on M can be represented by one of three equivalent objects:

(1) a bivector field Λ,
(2) a homomorphism $\widetilde{\Lambda}^1$ of vector bundles,
(3) a bilinear, skew-symmetric function $\widetilde{\Lambda}^2$.

The condition for Λ to be a Poisson structure is vanishing of the Schouten bracket: $[\Lambda, \Lambda] = 0$.

A skew-symmetric function

$$\lambda \colon \bigwedge^2 \mathsf{T}^* M \to \bigwedge^0 \mathsf{T}M \simeq M \times \mathbb{R}$$

defines a Poisson structure if the following Jacobi identity is fulfilled:

$$\lambda(df, \lambda(dg, dh)) + \lambda(dg, \lambda(dh, df)) + \lambda(dh, \lambda(df, dg)) = 0$$

where $f, g, h \in \Phi^0(\pi_M)$.

In order to get a definition of a Poisson structure in terms of a vector bundle morphism $\nu \colon \mathsf{T}^* M \to \mathsf{T}M$, we need the following theorem.

Theorem 4.4. A 2-vector field Λ on M defines Poisson structure if and only if

$$(\widetilde{\Lambda}^1)_* \Lambda_M \subset d_\mathsf{T} \Lambda$$

(one can say that Λ_M and $d_\mathsf{T}\Lambda$ are $\widetilde{\Lambda}^1$-related), where $\Lambda_M = (\Omega_M)^{-1}$ is the canonical 2-vector field on $\mathsf{T}^* M$.

Proof. In local coordinates $\Lambda = \frac{1}{2}\Lambda^{ij}\partial/\partial x^i \wedge \partial/\partial x^j$. The 2-vector field Λ defines a Poisson structure if and only if

$$\sum_i \left(\Lambda^{ij}\frac{\partial}{\partial x^i}\Lambda^{kl} + \Lambda^{ik}\frac{\partial}{\partial x^i}\Lambda^{lj} + \Lambda^{il}\frac{\partial}{\partial x^i}\Lambda^{jk} \right) = 0. \tag{4.4}$$

The canonical 2-vector field Λ_M on $\mathsf{T}^* M$ is given by

$$\Lambda_M = \sum_i \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial p_i}.$$

Moreover

$$d_\mathsf{T}\Lambda = \frac{1}{2}\sum_{ijk}\frac{\partial\Lambda^{ij}}{\partial x^k}\dot{x}^k\frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j} + \sum_{ij}\Lambda^{ij}\frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial x^j}. \tag{4.5}$$

For $a \in T^*M$,

$$\dot{x}^i((\widetilde{\Lambda}^1)(a)) = \sum_j \Lambda^{ji} p_i(a)$$

and, consequently,

$$(\widetilde{\Lambda}^1)_* \left(\frac{\partial}{\partial x^i} \right) = \frac{\partial}{\partial x^i} + \sum_{j,k} p_j \left(\frac{\partial}{\partial x^i} \Lambda^{jk} \right) \frac{\partial}{\partial \dot{x}^k}$$

$$(\widetilde{\Lambda}^1)_* \left(\frac{\partial}{\partial p_i} \right) = \sum_j \Lambda^{ij} \frac{\partial}{\partial \dot{x}^j}.$$

Hence

$$(\widetilde{\Lambda}^1)_* \Lambda_M = \sum_{i,j} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \dot{x}^j} + \sum_{ijk} p_l \Lambda^{ij} \left(\frac{\partial}{\partial x^i} \Lambda^{lk} \right) \frac{\partial}{\partial \dot{x}^k} \wedge \frac{\partial}{\partial \dot{x}^j}. \qquad (4.6)$$

It follows from (4.5) and (4.6) that $(\widetilde{\Lambda}^1)_* \Lambda_M \subset d_T\Lambda$ if and only if

$$\Lambda^{lk} p_l \frac{\partial}{\partial x^k} \Lambda^{ij} = \Lambda^{kj} \left(\frac{\partial}{\partial x^k} \Lambda^{li} \right) p_l - \Lambda^{ki} \left(\frac{\partial}{\partial x^k} \Lambda^{lj} \right) p_l$$

but this is equivalent to (4.4). $\qquad \square$

Now, we are ready for a proof of an analogue of theorem 4.3.

Theorem 4.5. A vector bundles morphism $\lambda \colon T^*M \to TM$ defines a Poisson structure on M if and only if the following diagram is commutative:

$$
\begin{array}{ccc}
\mathsf{TT}^*M & \xleftarrow{\quad\Lambda_M\quad} & \mathsf{T}^*\mathsf{T}^*M \\[4pt]
\mathsf{T}\lambda \downarrow & & \mathsf{T}^*\lambda \uparrow \\[4pt]
\mathsf{TT}M & & \mathsf{T}^*_{im(\lambda)}\mathsf{T}M \\[4pt]
\varepsilon'_M \uparrow & & \varepsilon_M \downarrow \\[4pt]
\mathsf{TT}M & \xleftarrow{\quad\mathsf{T}\lambda\quad} & \mathsf{TT}^*M
\end{array}
\qquad (4.7)
$$

Proof. Commutativity of the diagram is equivalent to

$$\mathsf{T}\lambda \circ \widetilde{\Lambda_M}^1 \circ \mathsf{T}^*\lambda = \varepsilon'_M \circ \mathsf{T}\lambda \circ \varepsilon_M$$

and to the dual equality

$$\mathsf{T}\lambda \circ (\widetilde{\Lambda_M}^1)^* \circ \mathsf{T}^*\lambda = \varepsilon'_M \circ \mathsf{T}\lambda^* \circ \varepsilon_M.$$

Since $(\widetilde{\Lambda_M}^1)^* = -\widetilde{\Lambda_M}^1$ we get $\mathsf{T}\lambda^* = -\mathsf{T}\lambda$ and, consequently, $\lambda^* = -\lambda$. It follows that there exists a bivector field Λ on M such that $\lambda = \widetilde{\Lambda}^1$. From theorem 4.4 we have that Λ is a Poisson structure if and only if the diagram (4.7) is commutative. $\qquad \square$

In the diagrams, $T^*\lambda$ is a relation (not a mapping) and diagrams are in the category of relations.

In order to illustrate the condition (4.7), let us consider the case of a linear Poisson structure. Let $M = V$ be a vector space. We have obvious identifications:

$$TV = V \times T_0 V = V \times V \qquad\qquad sT^*V = V \times V^*$$

$$TTV = (V \times V) \times (V \times V) \qquad\qquad TT^*V = (V \times V^*) \times (V \times V^*)$$

$$T^*T^*V = (V \times V^*) \times (V^* \times V).$$

With these identifications the canonical morphisms $\varepsilon_M, \kappa_M, \widetilde{\Lambda_M}^1$ look like the following:

$$\varepsilon_V: T^*TV \ni (v, w; a, b) \longmapsto (v, b; w, a) \in TT^*V$$

$$\kappa_V: TTV \ni (v, w; x, y) \longmapsto (v, x; w, y) \in TTV$$

$$\widetilde{\Lambda_V}^1: T^*T^*V \ni (v, a; w, b) \longmapsto (v, a; -w, b) \in TT^*V.$$

A linear Poisson structure Λ is given by a mapping

$$\widetilde{\Lambda}^1: T^*V \ni (v, a) \longmapsto (v, \lambda(v, a)) \in TV$$

where $\lambda: V \times V^* \to V$ is bilinear. For such Λ we have

$$T\widetilde{\Lambda}^1: (v, a; w, b) \longmapsto (v, \lambda(v, a); w, \lambda(v, b) + \lambda(w, a))$$

$$T^*\widetilde{\Lambda}^1: (v, \lambda(v, d); a, b) \longmapsto (v, d; a + {}^*\lambda(b, d), \lambda^*(v, b))$$

where ${}^*\lambda$ and λ^* are conjugate to λ with respect to the left and to the right argument respectively. The condition (4.7) reads as follows.

The mapping $\widetilde{\Lambda}^1$ defines a Poisson structure if and only if the following conditions are satisfied for each v, a, b:

(1) $\lambda(v, b) = -\lambda^*(v, b)$,
(2) $\lambda(v, {}^*\lambda(b, a)) = \lambda(\lambda(v, a), b) - \lambda(\lambda(v, b), a)$,

which means that

$${}^*\lambda: V^* \times V^* \to V^*$$

is a Lie bracket.

5. Tangent Poisson structures

Let (M, Λ) be a Poisson manifold, where M is a manifold and $\Lambda \in \Phi^2(\tau_M)$ satisfies $[\Lambda, \Lambda] = 0$. It follows from theorem 2.5 that $(TM, d_T\Lambda)$ is also a Poisson manifold:

$$[d_T\Lambda, d_T\Lambda] = d_T[\Lambda, \Lambda] = 0.$$

We call $d_T\Lambda$ the *tangent Poisson structure*. In local coordinates let

$$\Lambda = \frac{1}{2}\Lambda^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \qquad \Lambda^{ij} = -\Lambda^{ji}$$

then

$$d_T\Lambda = \Lambda^{ij}(x)\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial \dot{x}^j} + \frac{1}{2}\frac{\partial \Lambda^{ij}}{\partial x^k}\dot{x}^k\frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j}. \tag{5.1}$$

The Poisson structure Λ defines a Poisson bracket $\{f, g\}_\Lambda = \Lambda(df \wedge dg)$ which provides the algebra $\Phi^0(\tau_M)$ of smooth functions with a structure of Lie algebra. The tangent Poisson

structure defines a Poisson bracket $\{,\}_{d_T\Lambda}$ on TM which is characterized by the following relations:

$$\{d_T f, d_T g\}_{d_T\Lambda} = d_T\{f, g\}_\Lambda$$

$$\{d_T f, v_T g\}_{d_T\Lambda} = \{v_T f, d_T g\} = v_T\{f, g\}_\Lambda \tag{5.2}$$

$$\{v_T f, v_T g\}_{d_T\Lambda} = 0.$$

This is exactly the lift of Λ defined in [SdA, Co3]. For a 1-form $\mu \in \Phi^1(\pi_M)$ we put $\Lambda_\mu = i_\mu\Lambda$. For a function f the vector field Λ_{df} is the *Hamiltonian vector field generated by f*.

Theorem 5.1. The tangent Poisson structure $d_T\Lambda$ is linear with respect to the vector bundle structure in $\tau_M: TM \to M$. Moreover, for $\mu, \vartheta \in \Phi^1(\pi)$ we have the following formula:

$$\{i_T\mu, i_T\vartheta\}_{d_T\Lambda} = i_T(d\langle\Lambda, \mu \wedge \vartheta\rangle + i_{\Lambda_\mu}d\vartheta - i_{\Lambda_\vartheta}d\mu). \tag{5.3}$$

Proof. It is enough to prove formula (5.3). We have

$$\{i_T\mu, i_T\vartheta\}_{d_T\Lambda} = \langle d_T\Lambda, di_T\mu \wedge di_T\vartheta\rangle$$

$$= \langle d_T\Lambda, d_T\mu \wedge d_T\vartheta - d_T\mu \wedge i_T d\vartheta - i_T d\mu \wedge d_T\vartheta + i_T d\mu \wedge i_T d\vartheta\rangle. \tag{5.4}$$

Since the 1-forms $i_T d\mu$ and $i_T d\vartheta$ are vertical, we have

$$\langle d_T\Lambda, i_T d\mu \wedge i_T d\vartheta\rangle = 0.$$

Moreover

$$\langle d_T\Lambda, d_T\mu \wedge d_T\vartheta\rangle = i_{d_T\vartheta}i_{d_T\mu}d_T\Lambda = i_{d_T\vartheta}d_T(i_\mu\Lambda) = d_T(\langle\Lambda, \mu \wedge \vartheta\rangle)$$

and

$$\langle d_T\Lambda, d_T\mu \wedge i_T d\vartheta\rangle = \langle d_T(\Lambda_\mu), i_T d\vartheta\rangle = i_{d_T(\Lambda_\mu)}i_\Gamma v_T(d\vartheta)$$

$$= -i_\Gamma i_{d_T\Lambda_\mu}v_T d\vartheta = -i_\Gamma v_T(i_{\Lambda_\mu}d\vartheta) = -i_T(i_{\Lambda_\mu}d\vartheta)$$

where Γ is a second-order vector field (see section 2). Similarly

$$\langle d_T\Lambda, i_T d\mu \wedge d_T\vartheta\rangle = i_T(i_{\Lambda_\vartheta}\mu)$$

and the theorem follows. $\square$

Since $d_T\Lambda$ is a linear Poisson structure, it defines an algebroid structure in the dual vector bundle T^*M. The theorem provides an explicit formula for the Lie bracket in $\Phi^1(\pi_M)$:

$$\{\mu, \vartheta\}_\Lambda = d\langle\Lambda, \mu \wedge \vartheta\rangle + i_{\Lambda_\mu}\vartheta - i_{\Lambda_\vartheta}\mu$$

$$= \mathcal{L}_{\Lambda_\mu}\vartheta - \mathcal{L}_{\Lambda_\vartheta}\mu - d\langle\Lambda, \mu \wedge \vartheta\rangle.$$

This is exactly the well-known extension of a Poisson bracket to 1-forms. Thus we have obtained a new way of defining it using the tangent lift.

Example

Let us consider the Poisson structure on $\mathbb{R}^2$ given by

$$\Lambda = x^2 \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}.$$

The tangent lift of this structure is given by the following formula:

$$d_T\Lambda = x^2\left(\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \dot{y}}\right) + 2x\dot{x}\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}}.$$

For 1-forms $\mu = \mu_1(x,y)dx + \mu_2(x,y)dy$ and $\vartheta = \vartheta_1(x,y)dx + \vartheta_2(x,y)dy$ we have $i_T\mu = \mu_1(x,y)\dot{x} + \mu_2(x,y)\dot{y}$ and $i_T\vartheta = \vartheta_1(x,y)\dot{x} + \vartheta_2(x,y)\dot{y}$. We can easily calculate the Poisson bracket

$$\{i_T\mu, i_T\vartheta\}_{d_T\Lambda} = x^2\left[\mu_1\left(\frac{\partial\vartheta_1}{\partial y}\dot{x} + \frac{\partial\vartheta_2}{\partial y}\dot{y}\right) - \mu_2\left(\frac{\partial\vartheta_1}{\partial x}\dot{x} + \frac{\partial\vartheta_2}{\partial x}\dot{y}\right)\right.$$

$$\left. -\vartheta_1\left(\frac{\partial\mu_1}{\partial y}\dot{x} + \frac{\partial\mu_2}{\partial y}\dot{y}\right) + \vartheta_2\left(\frac{\partial\mu_1}{\partial x}\dot{x} + \frac{\partial\mu_2}{\partial x}\dot{y}\right)\right] + 2x\dot{x}(\mu_1\vartheta_2 - \mu_2\vartheta_1).$$

Hence, $\{i_T\mu, i_T\vartheta\}_{d_T\Lambda} = i_T\{\mu, \vartheta\}_\Lambda$, where

$$\{\mu, \vartheta\}_\Lambda = \left[x^2\left(\mu_1\frac{\partial\vartheta_1}{\partial y} - \mu_2\frac{\partial\vartheta_1}{\partial x} - \vartheta_1\frac{\partial\mu_1}{\partial y} + \vartheta_2\frac{\partial\mu_1}{\partial x}\right) + 2x(\mu_1\vartheta_2 - \mu_2\vartheta_1)\right]dx$$

$$+x^2\left(\mu_1\frac{\partial\vartheta_2}{\partial y} - \mu_2\frac{\partial\vartheta_2}{\partial x} - \vartheta_1\frac{\partial\mu_2}{\partial y} + \vartheta_2\frac{\partial\mu_2}{\partial x}\right)dy.$$

In particular, $\{xdy, dy\}_\Lambda = x^2 dy$ and $\{xdy, dx\}_\Lambda = -2x^2 dx$.

6. Canonical vector fields

It is well known [Tu2] that for a symplectic manifold (M, ω) the tangent structure $(TM, d_T\omega)$ is also a symplectic manifold. We use proposition 3.1, to get a simple proof that canonical vector fields on a symplectic manifold are Lagrangian submanifolds with respect to the tangent symplectic structure (compare with [SdA]). It justifies the concept of a generalized canonical system as a Lagrangian submanifold of the tangent Poisson manifold.

Proposition 6.1. Let $X: M \to TM$ be a vector field on a symplectic manifold (M, ω). The vector field X is canonical, i.e. $\pounds_X\omega = 0$ if and only if $X(M)$ is a Lagrangian submanifold of $(TM, d_T\omega)$.

Proof. From the proposition 3.1, $\pounds_X\omega = X^*d_T\omega$. Hence $\pounds_X\omega = 0$ if and only if $X^*d_T\omega = 0$, but the last means that $X(M)$ is isotropic in $(TM, d_T\omega)$ and, consequently, Lagrangian (the dimension of $X(M)$ is equal to the dimension of M).   □

We have an analogue of this theorem for Poisson manifolds. To formulate it we need the definition of a Lagrangian submanifold of a Poisson manifold.

Definition. Let (M, Λ) be a Poisson manifold. A submanifold $N \subset M$ is Lagrangian if for each $m \in N$

$$\tilde{\Lambda}^1(T_m N)^o = \tilde{\Lambda}^1(T_m^* M) \cap T_m N \tag{6.1}$$

Theorem 6.2. Let $X: M \to \mathsf{T}M$ be a vector field on a Poisson manifold (M, Λ). X is canonical, i.e. $\mathcal{L}_X \Lambda = 0$ if and only if $X(M)$ is a Lagrangian submanifold of $(\mathsf{T}M, \mathsf{d}_\mathsf{T}\Lambda)$.

Proof. From proposition 3.3, $\mathcal{L}_X \Lambda = 0$ if and only if, for $f, g \in C^\infty(M)$,

$$\langle \widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (X_* \mathsf{d}f), X_* \mathsf{d}g \rangle = 0.$$

On the other hand, $X(M)$ is Lagrangian if for each $v \in X(M)$

$$\{\widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (X_* \mathsf{d}f)(v) | f \in C^\infty(M)\} = \mathsf{T}_v X(M) \cap \widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (\mathsf{T}_v^* \mathsf{T}M). \tag{6.2}$$

Since $w \in \mathsf{T}_v X(M)$ is characterized by $X_* \mathsf{d}f(w) = 0$, the inclusion '$\subset$' is equivalent to

$$\langle \widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (X_* \mathsf{d}f)(v), X_* \mathsf{d}g \rangle = 0.$$

Thus, if $X(M)$ is Lagrangian then X is canonical. In order to prove that for a canonical X the submanifold $X(M)$ is Lagrangian, it is enough to show that the inclusion '$\supset$' holds. Let be $a \in \mathsf{T}_v^* \mathsf{T}M$. There are functions f, g on M such that $a = \mathsf{d}_v(\mathsf{d}_\mathsf{T} f) + \mathsf{d}_v(v_\mathsf{T} g)$. We then have (2.9)

$$\widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (a) = \mathsf{d}_\mathsf{T}(\widetilde{\Lambda}^1 \mathsf{d}f) + v_\mathsf{T}(\widetilde{\Lambda}^1 \mathsf{d}g). \tag{6.3}$$

It follows that

$$\mathsf{T}\tau_M(\widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 a) = \mathsf{T}\tau_M(\mathsf{d}_\mathsf{T}(\widetilde{\Lambda}^1 \mathsf{d}f) = \widetilde{\Lambda}^1 \mathsf{d}_m f \tag{6.4}$$

where $m = \tau_M v$. Thus the tangent projections of vectors in $\mathsf{T}X(M) \cap \widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (\mathsf{T}^* \mathsf{T}M)$ are in the image of $\widetilde{\Lambda}^1$. For a vector $w \in \widetilde{\Lambda}^1(\mathsf{T}^* M)$ there is only one vector $v \in \mathsf{T}X(M)$ such that $\mathsf{T}\tau_M v = w$, namely, $v = \mathsf{T}X(w)$. Let be $w = \widetilde{\Lambda}^1 \mathsf{d}_m f$. The inclusion '$\subset$' implies that $\widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (\mathsf{d}_{X(m)}(\mathsf{d}_\mathsf{T} f - v_\mathsf{T}(Xf))$ is tangent to $X(M)$. Since (6.3) and (6.4)

$$\mathsf{T}\tau_M \widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (\mathsf{d}(\mathsf{d}_\mathsf{T} f - v_\mathsf{T}(Xf))(X(m))) = \widetilde{\Lambda}^1 \mathsf{d}_m f = w$$

we have $\widetilde{\mathsf{d}_\mathsf{T}\Lambda}^1 (X_* \mathsf{d}f)(X(m)) = \mathsf{T}X(w)$. $\qquad\square$

7. Tangent Poisson–Lie structures

The growing interest in Poisson–Lie structures justifies analysis of every aspect of these structures. We show that the tangent lift of a Poisson–Lie structure is a Poisson–Lie structure (theorem 7.1). Vertical and complete lifts of left- (right-) invariant vector fields on G turn out to be left- (right-) invariant on $\mathsf{T}G$.

Let M, N be differentiable manifolds. Since there is a canonical identification $\mathsf{T}(M \times N) \simeq \mathsf{T}M \times \mathsf{T}N$, we have also canonical inclusions $\bigwedge^2 \mathsf{T}M \times \bigwedge^2 \mathsf{T}N \subset \bigwedge^2 \mathsf{T}(M \times N)$ and $\Phi^2(\tau_M) \oplus \Phi^2(\tau_N) \subset \Phi^2(\tau_{N \times M})$. It is trivial task to verify, that if (M, Λ) and (N, Π) are Poisson manifolds then $(M \times N, \Lambda \oplus \Pi)$ is also a Poisson manifold. The Poisson structure $\Lambda \oplus \Pi$ is called the *product Poisson structure*. Finally, we recall the notion of a *Poisson map*. A smooth mapping $\varphi: M \to N$ is a Poisson map if

$$\bigwedge^2 \mathsf{T}\varphi \circ \Lambda = \Pi \circ \varphi$$

or, equivalently,

$$\mathsf{T}\varphi \circ \widetilde{\Lambda} \circ \mathsf{T}^*\varphi = \widetilde{\Pi}$$

where $\widetilde{\Lambda} = \widetilde{\Lambda}^1$, $\widetilde{\Pi} = \widetilde{\Pi}^1$.

Let (G, m) be a Lie group and let (G, Λ) be a Poisson manifold. We say that (G, m, Λ) is a *Poisson–Lie group* if the group multiplication m is a Poisson mapping:

$$m: (G \times G, \Lambda \oplus \Lambda) \to (G, \Lambda)$$

or, equivalently,

$$\mathsf{T}m \circ (\widetilde{\Lambda} \times \widetilde{\Lambda}) \circ \mathsf{T}^*m = \widetilde{\Lambda}. \tag{7.1}$$

We say in this case that Λ is *multiplicative*.

Applying the tangent functor to (7.1), we get

$$\mathsf{T}\mathsf{T}m \circ (\mathsf{T}\widetilde{\Lambda} \times \mathsf{T}\widetilde{\Lambda}) \circ \mathsf{T}\mathsf{T}^*m = \mathsf{T}\widetilde{\Lambda}. \tag{7.2}$$

It is well known that $(\mathsf{T}G, \mathsf{T}m)$ is a Lie group.

Theorem 7.1. Let (G, m, Λ) be a Poisson–Lie group. Then $(\mathsf{T}G, \mathsf{T}m, \mathsf{d}_\mathsf{T}\Lambda)$ is also a Poisson–Lie group.

Proof. We have to show that $\mathsf{d}_\mathsf{T}\Lambda$ is multiplicative with respect to $\mathsf{T}m$, i.e. that

$$\mathsf{T}\mathsf{T}m \circ (\widetilde{\mathsf{d}_\mathsf{T}\Lambda} \times \widetilde{\mathsf{d}_\mathsf{T}\Lambda}) \circ \mathsf{T}^*\mathsf{T}m = \widetilde{\mathsf{d}_\mathsf{T}\Lambda}.$$

Since, due to (2.6), $\widetilde{\mathsf{d}_\mathsf{T}\Lambda} = \kappa_G \circ \mathsf{T}\widetilde{\Lambda} \circ \varepsilon_G$, it follows from functorial properties of κ and ε that

$$\mathsf{T}\mathsf{T}m \circ (\widetilde{\mathsf{d}_\mathsf{T}\Lambda} \times \widetilde{\mathsf{d}_\mathsf{T}\Lambda}) \circ \mathsf{T}^*\mathsf{T}m = \mathsf{T}\mathsf{T}m \circ \kappa_{G\times G} \circ (\mathsf{T}\widetilde{\Lambda} \times \mathsf{T}\widetilde{\Lambda}) \circ \varepsilon_{G\times G} \circ \mathsf{T}^*\mathsf{T}m$$

$$= \kappa_G \circ \mathsf{T}\mathsf{T}m \circ (\mathsf{T}\widetilde{\Lambda} \times \widetilde{\Lambda}) \circ \mathsf{T}\mathsf{T}^*m \circ \varepsilon_G$$

and by (7.2) the required identity follows.   □

Let $\mathfrak{g}$ be the Lie algebra of the Lie group G. The tangent bundle $\mathsf{T}G$ can be trivialized by right or left translations:

$$(K^r, \tau_G): \mathsf{T}G \to \mathfrak{g} \times G$$

or

$$(\tau_G, K^l): \mathsf{T}G \to G \times \mathfrak{g}$$

where $K^r(K^l)$ is the right-invariant (left-invariant) Maurer–Cartan form. The group structure in $\mathsf{T}G$ is given by the formula

$$(X, g) \cdot (Y, h) = (X + \mathrm{Ad}(g)Y, gh)$$

in the right trivialization and

$$(g, X) \cdot (h, Y) = (gh, X + \mathrm{Ad}(h^{-1})Y)$$

in the left trivialization. The neutral element e^T is represented by $(0, e)$ in the right trivialization and by $(e, 0)$ in the left trivialization.

The Lie algebra $\mathsf{T}\mathfrak{g}$ of $\mathsf{T}G$ is isomorphic as a linear space to $\mathfrak{g} \times \mathfrak{g}$. This isomorphism is implemented by the zero section $j_G: G \to \mathsf{T}G$ and the obvious embedding $j_\mathfrak{g}: \mathfrak{g} \to \mathsf{T}_e G \subset \mathsf{T}G$ and is given by the following formula:

$$\mathfrak{g} \times \mathfrak{g} \ni (X, Y) \to \mathsf{T}_e j_G(X) + \mathsf{T}_0 j_\mathfrak{g}(Y) \in \mathsf{T}_{e^\mathsf{T}}\mathsf{T}G. \tag{7.3}$$

From now on we shall denote $T_e j_G(X)$ by $\dot{X}$ and $T_0 j_\vartheta(Y)$ by $\widehat{Y}$. We have the following commutation rules:

$$[\dot{X}, \dot{Y}]_{T\vartheta} = [X, Y]_\vartheta$$

$$[\dot{X}, \widehat{Y}]_{T\vartheta} = [\widehat{X}, \dot{Y}]_{T\vartheta} = \widehat{[X, Y]}_\vartheta$$

$$[\widehat{X}, \widehat{Y}]_{T\vartheta} = 0.$$

It follows that the Lie algebra $T\vartheta$ is a semidirect product $\mathfrak{g} \ltimes \mathfrak{g}$ with respect to the adjoint representation in $\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$, i.e.

$$[(X, Y), (X', Y')]_{\vartheta \times \vartheta} = ([X, X']_\vartheta, [X, Y']_\vartheta + [Y, X']_\vartheta).$$

Theorem 7.2. Let be $X \in \mathfrak{g}$ and let X_G^l be the corresponding left-invariant vector field on G. Then

$$\widehat{X}_{TG}^l = (X_G^l)^v \qquad \text{and} \qquad \dot{X}_{TG}^l = (X_G^l)^c$$

i.e. the corresponding to $\widehat{X}$ and $\dot{X}$ left-invariant vector fields on TG are the vertical and complete lifts of X_G^l respectively. An analogous statement is true for right-invariant vector fields.

Proof. The group multiplication Tm in TG is the tangent lift of the group multiplication m in G. It follows that

$$Tm(u_g, v_h) = TL_g(v_h) + TR_h(u_g)$$

where L_g and R_h are left and right translations by g and h respectively. The left translation $L_{u_g}^\mathsf{T}$ by u_g in TG is therefore given by

$$L_{u_g}^\mathsf{T}(v_h) = TL_g(v_h) + TR_h(u_g).$$

It is easy to verify that $(X_G^l)^c(e^T) = \dot{X}$ and $(X_G^l)^v(e^T)$. Since the mapping (7.3) is a linear isomorphism, it is enough to show that vector fields $(X_G^l)^c$ and $(X_G^l)^v$ are left-invariant on TG. In other words, we have to show that flows they generate commute with left translations.

The flow ψ^t of the vertical lift is given by

$$\psi^t(v_h) = v_h + t X_G^l(h)$$

we then have

$$L_{u_g}^\mathsf{T} \circ \psi^t(v_h) = TL_g(v_h + t X_G^l(h)) + TR_h(u_g)$$

$$= TL_g(v_h) + TR_h(u_g) + t TL_g(X_G^l(h))$$

$$= TL_g(v_h) + TR_h(u_g) + t X_G^l(gh) = \psi^t \circ L_{u_g}^\mathsf{T}(v_h).$$

We made use of the left-invariance of X_G^l.

The flow of the complete lift $(X_G^l)^c$ is the tangent lift of the flow φ^t of X_G^l. We have

$$L_{u_g}^\mathsf{T} \circ T\varphi^t(v_h) = TL_g \circ T\varphi^t(v_h) + TR_{\varphi^t(h)}(u_g)$$

$$= T(L_g \circ \varphi^t)(v_h) + TR_{\varphi^t(h)}(u_g)$$

and, since φ^t is the flow of a left-invariant vector field, i.e.

$$L_g \circ \varphi^t = \varphi^t \circ L_g$$

and

$$\varphi^t \circ R_h(g) = \varphi^t(gh) = L_g(\varphi^t(h)) = R_{\varphi^t(h)}(g)$$

we get

$$T(L_g \circ \varphi^t)(v_h) + TR_{\varphi^t(h)}(u_g) = T(\varphi^t \circ L_g)(v_h) + T(\varphi^t(h) \circ R_h)(u_g) = T\varphi^t \circ L^T_{u_g}(v_h).$$

$\square$

8. Tangent Poisson–Lie algebras of Poisson–Lie groups

In this section we show that the tangent to the Poisson–Lie algebra of a Poisson–Lie group (G, Λ) is the Poisson–Lie algebra of the Poisson–Lie group $(TG, d_T\Lambda)$. A special case of of Poisson structures defined by r-matrices is discussed.

The Lie algebra of a Poisson–Lie group inherits a Poisson structure. We recall here a standard construction. More natural and more geometric one will be given in section 10 (proposition 10.4).

Let (G, m, Λ) be a Poisson–Lie group. A Poisson structure Λ on a Lie group can be regarded, using the right trivialization of TG, as a mapping $\bar{\Lambda}: G \to \mathfrak{g} \wedge \mathfrak{g}$. The Poisson structure Λ is multiplicative if and only if $\bar{\Lambda}$ is a 1-cocycle of G with respect to the adjoint representation of G in $\mathfrak{g} \wedge \mathfrak{g}$. In particular, we have $\bar{\Lambda}(e) = 0$. The tangent mapping

$$\lambda = T_e\bar{\Lambda}: T_eG = \mathfrak{g} \to T_0(\mathfrak{g} \wedge \mathfrak{g}) = \mathfrak{g} \wedge \mathfrak{g}$$

being a 1-cocycle of $\mathfrak{g}$, defines a cobracket (Poisson bracket on $\mathfrak{g}$). The pair $(\mathfrak{g}, \lambda)$ is called the *tangent Poisson–Lie algebra* of the Poisson–Lie group (G, m, Λ).

Let $(X_1, \ldots, X_n)$ be a basis of the Lie algebra $\mathfrak{g}$. We can write

$$\Lambda = \sum \lambda^{ij} X^r_i \wedge X^r_j$$

where λ^{ij} are smooth functions on G and X^r_i are the corresponding right-invariant vector fields. We then have

$$\bar{\Lambda}(g) = \sum \lambda^{ij}(g) X_i \wedge X_j$$

and

$$\lambda(X_k) = \langle X_k, d\lambda^{ij}(e)\rangle X_i \wedge X_j = \tfrac{1}{2} c^{ij}_k X_i \wedge X_j$$

where c^{ij}_k are structure constants of the Lie algebra $\mathfrak{g}$. The cobracket $\lambda: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ may be regarded as a bivector field on $\mathfrak{g}$ which defines a linear Poisson structure on $\mathfrak{g}$.

Definition. A Poisson–Lie algebra $(\mathfrak{g}, \delta)$ is a Lie algebra $\mathfrak{g}$ and a 1-cocycle (cobracket) $\delta: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ (with respect to the adjoint representation of $\mathfrak{g}$ in $\mathfrak{g} \wedge \mathfrak{g}$) such that it defines a Poisson structure on $\mathfrak{g}$ or, equivalently, that the dual mapping $\delta^*: \mathfrak{g}^* \wedge \mathfrak{g}^*$ is a Lie bracket on $\mathfrak{g}^*$.

The tangent Poisson–Lie algebra is an example of an abstract Poisson–Lie algebra. For simple connected Lie groups there is one-to-one correspondence between Poisson–Lie structures on G and Poisson–Lie algebra structures on $\mathfrak{g}$. The Poisson–Lie algebra $(\mathfrak{g}, \lambda)$ can be seen as a quadruple $(\mathfrak{g}, \mathfrak{g}^*, [\cdot, \cdot], [\cdot, \cdot]^*)$ and, for this reason, it is called sometimes a *Lie bialgebra*.

Theorem 8.1. Let $\lambda: \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ be the cobracket of the tangent Poisson–Lie algebra of a Poisson–Lie group (G, m, Λ). Then

$$d_T\lambda: T\mathfrak{g} \to T\mathfrak{g} \wedge T\mathfrak{g}$$

is the cobracket of the tangent Poisson–Lie algebra of the tangent Poisson–Lie group $(TG, Tm, d_T\Lambda)$. The dual mapping

$$(d_T\lambda)^*: (T\mathfrak{g})^* \wedge (T\mathfrak{g})^* \to (T\mathfrak{g})^*$$

is the Lie bracket on the tangent Lie algebra of $(\mathfrak{g}^*, \lambda^*)$.

Proof. For any $X \in \Phi^r(\tau_M)$ and for any smooth function $f \in \Phi^0(\pi_M)$ we have

$$d_T(fX) = \tau_M^*(f)d_T X + d_T f \cdot v_T(X).$$

Moreover, $Y^v(\tau_M^* f) = 0$, $Y^v(d_T f) = Y^c(\tau_M^* f) = \tau_M^*(Y(f))$ and $Y^c(d_T f) = d_T(Y(f))$ for any vector field $Y \in \Phi^1(\tau_M)$.

Now, let $\Lambda = \Lambda^{ij} X_i^r \wedge X_j^r$ and $\lambda(X_k) = \frac{1}{2}c_k^{ij} X_i \wedge X_j$ for a basis $(X_1, \ldots, X_n)$ in $\mathfrak{g}$. We have

$$d_T\Lambda = 2\tau_G^*(\Lambda^{ij})(X_i^r)^c \wedge (X_j^r)^v + d_T(\Lambda^{ij})(X_i^r)^v \wedge (X_j^r)^v$$

$$= 2\tau_G^*(\Lambda^{ij})(\dot{X}_i)^r \wedge (\widehat{X}_j)^r + d_T(\Lambda^{ij})(\widehat{X}_i)^r \wedge (\widehat{X}_j)^r.$$

The cobracket δ on the Lie algebra of TG is given by the formulae

$$\delta(\widehat{X}_k) = (X_k^r)^v(2\tau_G^*(\Lambda^{ij}))(e^T)\dot{X}_i \wedge \widehat{X}_j + (X_k^r)^v(d_T(\Lambda^{ij}))(e^T)\widehat{X}_i \wedge \widehat{X}_j$$

$$= \tau_G^*(X_k^r(\Lambda^{ij}))(e^T)\widehat{X}_i \wedge \widehat{X}_j = X_k^r(\Lambda^{ij})(e)\widehat{X}_i \wedge \widehat{X}_j$$

$$= \langle X_k, d\Lambda^{ij}\rangle(e)\widehat{X}_i \wedge \widehat{X}_j = \frac{1}{2}c_k^{ij}\widehat{X}_i \wedge \widehat{X}_j$$

and

$$\delta(\dot{X}_k) = (X_k^r)^c(2\tau_G^*(\Lambda^{ij}))(e^T)\dot{X}_i \wedge \widehat{X}_j + (X_k^r)^c(d_T(\Lambda^{ij}))(e^T)\widehat{X}_i \wedge \widehat{X}_j$$

$$= 2X_k^r(\Lambda^{ij})(e)\dot{X}_i \wedge \widehat{X}_j + d_T(X_j^r(\Lambda^{ij}))(e^T)\widehat{X}_i \wedge \widehat{X}_j$$

$$= c_k^{ij}\dot{X}_i \wedge \widehat{X}_j$$

($d_T(f)$ is zero on the zero section). It follows that $\delta = d_T\lambda$. $\qquad\square$

Let (G, m, Λ) be a Poisson–Lie group and let $(\mathfrak{g}, [,], \mathfrak{p})$ be its Lie bialgebra. Let us suppose that $\mathfrak{p}$ is a coboundary (e.g. G is semisimple), i.e. that

$$\mathfrak{p}(X) = [X, \mathfrak{r}] = r^{ij}([X, X_i] \wedge X_j + X_i \wedge [X, X_j])$$

for some $\mathfrak{r} = r^{ij} X_i \wedge X_j \in \mathfrak{g} \wedge \mathfrak{g}$. It is known [Dr] that in this case the Poisson structure Λ on G can be written in the form

$$\Lambda = \mathfrak{r}^l - \mathfrak{r}^r$$

where $\mathfrak{r}^l$ and $\mathfrak{r}^r$ are the left- and right-invariant 2-vector fields corresponding to $\mathfrak{r}$. Since Λ is a Poisson structure, r must satisfy the generalized classical Yang–Baxter equation

$$ad_X[\mathfrak{r}, \mathfrak{r}] = 0$$

for every $X \in \mathfrak{g}$. The bracket $[\mathfrak{r}, \mathfrak{r}]$ is the algebraic Schouten bracket. An element of $\mathfrak{g} \wedge \mathfrak{g}$ which satisfies this equation is called a *generalized r-matrix* and the corresponding Poisson structure Λ is called *quasitriangular*.

Theorem 8.2. Let $\Lambda = r^l - r^r$ be a quasitriangular Poisson–Lie structure on a Lie group G with the r-matrix $r = r^{ij} X_i \wedge X_j$, $r^{ij} = -r^{ji}$. Then $d_T \Lambda$ is a quasitriangular Poisson–Lie structure on TG with the r-matrix $d_T r = 2r^{ij} \dot{X}_i \wedge \widehat{X}_j$.

Proof. Since $\Lambda = r^{ij}(X_i^l \wedge X_j^l - X_i^r \wedge X_j^r)$, we have

$$d_T \Lambda = 2r^{ij}((X_i^l)^c \wedge (X_j^l)^v - (X_i^r)^c \wedge (X_j^r)^v)$$

$$= 2r^{ij}((\dot{X}_i)^l \wedge (\widehat{X}_j)^l - (\dot{X}_i)^r \wedge (\widehat{X}_j)^r) = (d_T r)^l - (d_T r)^r$$

and it is easy to check that $d_T r$ is really an r-matrix.   □

9. Tangent lifts of generalized foliations

Symplectic foliations of Poisson manifolds are important examples of generalized foliations. In this section we define the tangent lift of a generalized foliation and discuss its basic properties.

Definition. A *generalized distribution* on a manifold M is a subset $S \subset TM$ such that $S(x) = S \cap T_x M$ is a linear subspace for each point $x \in M$. S is said to be *smooth* if it is generated by the family

$$\mathcal{X}(S) = \{X \in \Phi^1(\tau_M) : \forall x \in M \ \ X(x) \in S(x)\}$$

of smooth vector fields, i.e. $S(x)$ is spanned by $\{X(x) : X \in \mathcal{X}(S)\}$.

A smooth distribution is *completely integrable* if for every point $x \in M$ there exists an integral submanifold of S, everywhere of maximal dimension, which contains x.

The maximal integral submanifolds of a completely integrable distribution S form a partition of M, called the *generalized foliation* of M defined by S.

Let us notice that for a completely integrable distribution S the family $\mathcal{X}(S)$ is a Lie subalgebra of the Lie algebra of vector fields on M. The following theorem is due to Sussmann [Sus].

Theorem (Sussmann). Let S be a smooth distribution on M and let $\mathcal{D} \subset \mathcal{X}(S)$ be a family of vector fields such that $\mathcal{D}$ spans S. The following properties are equivalent

(1) S is completely integrable,
(2) S is invariant with respect to flows $\exp(tX)$ of vector fields $X \in \mathcal{D}$,
(3) flows of vector fields from $\mathcal{X}(S)$ preserve S.

Theorem 9.1. Let S be a completely integrable generalized distribution. Then the distribution S^T generated by the family $\{X^v, X^c : X \in \mathcal{X}(S)\}$ of vector fields on TM is completely integrable

Proof. We have $\exp(tX^v)(v) = v + tX(\tau_M v)$ and $\exp(tX^c) = T\exp(tX)$. Due to the formulae

$$(\exp(tX^v))_* Y^v = Y^v \qquad (\exp(tX^v))_* Y^c = Y^c + t[X, Y]^v$$

and

$$(\exp(tX^c))_* Y^v = ((\exp(tX))_* Y)^v \qquad (\exp(tX^c))_* Y^c = ((\exp(tX))_* Y)^c$$

it follows that S^T is invariant with respect to flows of X^c and X^v. From the theorem of Sussmann, S^T is integrable.   □

Definition. The *tangent foliation* $\mathcal{F}^{\mathsf{T}}$ of a generalized foliation $\mathcal{F}$ defined by S is the foliation defined by S^{T}.

Example

Let us consider the distribution S on $\mathbb{R}$, generated by vector fields vanishing at $0 \in \mathbb{R}$. The corresponding foliation is as follows:

$$\mathcal{F} = \{\{0\}, \mathbb{R}_+, \mathbb{R}_-\} \qquad \text{where} \quad \mathbb{R}_\pm = \{x \in \mathbb{R} : \pm x > 0\}.$$

We identify $\mathsf{T}\mathbb{R}$ with $\mathbb{R}^2$ (with coordinates (x, y)) and we get that vertical and complete lifts of vector fields from $\mathcal{X}(S)$ are of the form

$$f(x)\frac{\partial}{\partial y}$$

and

$$f(x)\frac{\partial}{\partial x} + f'(x)y\frac{\partial}{\partial y}$$

where $f \in C^\infty(\mathbb{R})$ vanishes at 0.

These vector fields generate the distribution S^{T} with $S^{\mathsf{T}}(x, y) = \text{span}\,(\partial/\partial x, \partial/\partial y)$ if $x \neq 0$, $S^{\mathsf{T}}(0, y) = \text{span}\,(\partial/\partial y)$ if $y \neq 0$, and $S^{\mathsf{T}}(0, 0) = \{0\}$. Hence, the corresponding tangent foliation $\mathcal{F}^{\mathsf{T}}$ consists of two half-plains $P_\pm = \{(x, y) : \pm x > 0\}$, two half-lines $L_\pm = \{(0, y) : \pm y > 0\}$ and the point $(0, 0)$ as a zero-dimensional leaf.

Proposition 9.2. If a 1-form μ annihilates a completely integrable distribution S then $\mathsf{v}_{\mathsf{T}}(\mu)$ and $\mathsf{d}_{\mathsf{T}}(\mu)$ annihilate S^{T}.

Proof. Let $X \in \mathcal{X}(S)$. We have from proposition (2.6) that

$$\langle \mathsf{v}_{\mathsf{T}}(\mu), X^v \rangle = 0 \qquad \langle \mathsf{d}_{\mathsf{T}}(\mu), X^c \rangle = \mathsf{d}_{\mathsf{T}}\langle \mu, X \rangle = 0$$

and

$$\langle \mathsf{v}_{\mathsf{T}}(\mu), X^c \rangle = \langle \mathsf{d}_{\mathsf{T}}(\mu), X^v \rangle = \mathsf{v}_{\mathsf{T}}(\langle \mu, X \rangle) = 0.$$

$\square$

Proposition 9.3. If a submanifold $N \subset M$ is a union of leaves of the foliation $\mathcal{F}$ (N is $\mathcal{F}$-foliated) then $\mathsf{T}_N M = \tau_M^{-1}(N)$ is $\mathcal{F}^{\mathsf{T}}$-foliated.

Proof. It is enough to prove proposition in the case of N being a single leaf. Let F be a leaf of $\mathcal{F}^{\mathsf{T}}$. Since $\mathsf{T}\tau_M(X^v) = 0$ and $\mathsf{T}\tau_M(X^c) = X$ for any vector field X on M, the tangent projection $\tau_M(F)$ of F is contained in a leaf of $\mathcal{F}$. It follows that $\mathsf{T}_N M$ is $\mathcal{F}^{\mathsf{T}}$-foliated. $\square$

Proposition 9.4. If F is a leaf of $\mathcal{F}$ then $\mathsf{T}F$ is a leaf of $\mathcal{F}^{\mathsf{T}}$.

Proof. It is obvious that $\mathsf{T}F \subset S^{\mathsf{T}}$ where S is the generalized distribution related to $\mathcal{F}$. We have to show that $\mathsf{T}F$ is maximal. Since F is maximal, $\mathsf{T}_x F = S(x)$ and $S(F)$ is spanned by vector fields tangent to F. On the other hand it is easy to see that if X is a vector field on M, tangent to F on F, then $\mathsf{d}_{\mathsf{T}}X$ and $\mathsf{v}_{\mathsf{T}}X$ are tangent to $\mathsf{T}F$ on $\mathsf{T}F$. Since S^{T} is generated by the family $\{X^v, X^c : X \in \mathcal{X}(S)\}$, $S^{\mathsf{T}}(\mathsf{T}F) = \mathsf{T}\mathsf{T}F$, i.e. $\mathsf{T}F$ is an integral submanifold of S^{T} which is clearly maximal. $\square$

10. Symplectic foliations of Poisson manifolds

Let (M, Λ) be a Poisson manifold. The characteristic distribution S of Λ is generated by Hamiltonian vector fields. It is well known that S is completely integrable and that Λ defines symplectic structures on leaves of S.

Proposition 10.1. S^{T} is the characteristic distribution of $d_{\mathsf{T}}\Lambda$.

Proof.   Since the vertical and tangent lifts of 1-forms on M generate the module of 1-forms on $\mathsf{T}M$, it is enough to notice that (2.9) implies

$$\langle v_{\mathsf{T}}(\mu), d_{\mathsf{T}}\Lambda \rangle = (i_\mu \Lambda)^v \qquad \text{and} \qquad \langle d_{\mathsf{T}}\mu, d_{\mathsf{T}}\Lambda \rangle = (i_\mu \Lambda)^c$$

and that, consequently, the characteristic distribution of $d_{\mathsf{T}}\Lambda$ is generated by the complete and vertical lifts of Hamiltonian vector fields on (M, Λ), i.e. of vector field from $\mathcal{X}(S)$. $\square$

The following theorem by Weinstein [We] describes the local structure of a Poisson manifold.

Theorem (Weinstein).   Let (M, Λ) be a Poisson manifold of rank $2k$ at $x_0 \in M$. Then there is an open neighbourhood U of x_0 such that $(U, \lambda|_U)$ is isomorphic to a product $(N \times V, \Lambda_N \times \Lambda_V)$ of Poisson manifolds where (N, Λ_N) is a symplectic manifold of dimension $2k$ and the rank of (V, Λ_V) is zero at z_0, $x_0 = (y_0, z_0)$.

The theorem of Weinstein shows that while analysing only local properties of Poisson manifolds it is enough to consider two cases:

(1) Λ is regular,
(2) Λ vanishes at a point.

Theorem 10.2.   Let (M, Λ) be a Poisson manifold.

(1) If Λ is a regular Poisson structure of rank $2k$ then $d_{\mathsf{T}}\Lambda$ is regular of rank $4k$.
(2) If Λ is of rank 0 at $x_0 \in M$ then $\mathsf{T}_{x_0}M$ is a Poisson submanifold of $(\mathsf{T}M, d_{\mathsf{T}}\Lambda)$ and $d_{\mathsf{T}}\Lambda$ defines on $\mathsf{T}_{x_0}M$ a linear Poisson structure (Kostant–Kirillov–Souriau structure). It induces then a Lie algebra structure on $\mathsf{T}^*_{x_0}M$.

Proof.

(1) It follows from the theorem of Weinstein that we can choose local coordinates on M such that

$$\Lambda = \sum_{j,i=1}^{r} \Lambda^{ij} \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \qquad \Lambda^{ij} = -\Lambda^{ji} \qquad \det(\Lambda^{ij}) \neq 0$$

where Λ^{ij} are constant. Hence, in the adopted coordinate system,

$$d_{\mathsf{T}}\Lambda = 2 \sum_{j,i=1}^{r} \Lambda^{ij} \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial x^j}.$$

(2) We have locally

$$\Lambda(x) = \sum_{j,i} \Lambda^{ij}(x) \frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} \qquad \Lambda^{ij} = -\Lambda^{ji} \qquad \Lambda^{ij}(x_0) = 0$$

and

$$d_T \Lambda(x_0, \dot{x}) = \sum_{j,i,k} \frac{\partial \Lambda^{ij}}{\partial x^k}(x_0)\dot{x}^k \frac{\partial}{\partial \dot{x}^i} \wedge \frac{\partial}{\partial \dot{x}^j}.$$

Hence $T_{x_0}M$ is a Poisson submanifold with the Poisson bracket

$$\{\dot{x}^i, \dot{x}^j\} = 2\sum_k \frac{\partial \Lambda^{ij}}{\partial x^k}(x_0)\dot{x}^k.$$

$\square$

Corollary. If Λ is of rank 0 at x_0 then the Poisson structure $d_T\Lambda$ on T_{x_0} is given by the formula

$$d_T\Lambda(v) = v_T(\mathcal{L}_{\tilde{v}}\Lambda)(v) \qquad v \in T_{x_0}M$$

where $\tilde{v} \in \Phi^1(\tau_M)$ is such that $\tilde{v}(x_0) = v$.

Proof. Since $\Lambda(x_0) = 0$ it follows that $d_T\Lambda(v)$ is vertical and, consequently,

$$v_T(\tilde{v}^* d_T\Lambda)(v) = d_T\Lambda(v)$$

(see section 3). It follows from theorem 3.2 that

$$d_T\Lambda(v) = v_T(\mathcal{L}_{\tilde{v}}\Lambda)(v).$$

$\square$

The following two propositions complete our discussion on the structure of the tangent Poisson manifold.

Proposition 10.3. Let f be a local Casimir of Λ, i.e. $\Lambda(df, \cdot) = 0$. Then $v_T(f)$ and $d_T(f)$ are local Casimirs of $d_T\Lambda$.

Moreover, if Λ is regular at $x \in M$ with symplectic leaves determined by Casimirs $(f_1, \ldots, f_n)$, then Λ is regular at $v \in T_xM$ with symplectic leaves determined by Casimirs $(\tau_M^*(f_1), \ldots, \tau_M^*(f_n), d_T f_1, \ldots, d_T f_n)$.

Proof. It follows from (2.9) that if f generates a zero-Hamiltonian field then also $\tau_M^* f$ and $d_T f$ generate the zero-Hamiltonian field on TM, i.e. they are Casimirs.

Let $x \in M$ be a regular point of Λ with the symplectic foliation in the neighbourhood of x defined by Casimirs $(f_1, \ldots, f_n)$. We may assume that $df_1, \ldots, df_n$ are linearly independent at x. It follows that $(d\tau_M^*(f_1), \ldots, d\tau_M^*(f_n), dd_T f_1, \ldots, dd_T f_n)$ are linearly independent at $v \in T_xM$. Since the rank of $d_T\Lambda$ is $2(\dim M - n)$ (theorem 10.2) the theorem follows. $\square$

If (G, Λ) is a Poisson–Lie group then $\Lambda(e) = 0$ and identifying $\mathfrak{g}$ with T_eG we get a Poisson structure on $\mathfrak{g}$ induced by $d_T\Lambda$.

Proposition 10.4. The Poisson structure on $\mathfrak{g}$ induced by $d_T\Lambda$ is equal to the Poisson structure defined by the cobracket of the tangent Poisson–Lie algebra.

Proof.   Let (x^i) be a coordinate system on G centered at e, i.e $x^i(e) = 0$. It defines a basis (X_i) of the Lie algebra $\mathfrak{g}$, $X_i = \partial(e)/\partial x^i$. There are functions a_i^k on G such that

$$\frac{\partial}{\partial x_i} - X_i^r = a_i^k X_i^r$$

where $a_i^k(0) = 0$. We then have

$$\Lambda = \Lambda^{ij}\frac{\partial}{\partial x^i} \wedge \frac{\partial}{\partial x^j} = \lambda^{ij} X_i^r \wedge X_j^r$$

and

$$\Lambda^{ij}(0) = \lambda^{ij}(0).$$

The induced cobracket λ is given by

$$\lambda(X_k) = \frac{\partial \lambda^{ij}}{\partial x_k}(0) X_i \wedge X_j.$$

On the other hand,

$$\mathrm{d}_\mathrm{T}|_{\mathrm{T}_e G} = \frac{\partial \Lambda^{ij}}{\partial x_k}(0)\frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial x_j} = \frac{\partial \Lambda^{ij}}{\partial x_k}(0) X_i \wedge X_j.$$

It follows that

$$\lambda^{ij} = \Lambda^{ij} + \Lambda^{il}a_l^j - \Lambda^{jl}a_l^i + \tfrac{1}{2}(a_k^i a_l^j - a_k^j a_l^i)\Lambda^{kl}.$$

Hence

$$\frac{\partial \lambda^{ij}}{\partial x_k}(0) = \frac{\partial \Lambda^{ij}}{\partial x_k}(0).$$

$\square$

11. Examples

Example 1

On $\mathfrak{su}(2)^* \simeq \mathbb{R}^3$ consider the linear Poisson structure

$$\Lambda = z\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} + x\frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z} + y\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial x}.$$

The symplectic foliation of $\mathbb{R}^3$ is the union of two-dimensional spheres $x^2 + y^2 + z^2 = r > 0$ and the origin $(0, 0, 0)$ as a zero-dimensional leaf. It is regular outside the origin and is defined by the Casimir $f(x, y, z) = x^2 + y^2 + z^2$.

The tangent Poisson structure is given by the formula

$$\mathrm{d}_\mathrm{T}\Lambda = z\left(\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial y} + \frac{\partial}{\partial x} \wedge \frac{\partial}{\partial \dot{y}}\right) + x\left(\frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial z} + \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial \dot{z}}\right)$$

$$+ y\left(\frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial x} + \frac{\partial}{\partial z} \wedge \frac{\partial}{\partial \dot{x}}\right) + \dot{z}\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}} + \dot{x}\frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial \dot{z}} + \dot{y}\frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial \dot{x}}.$$

The symplectic foliation of $\mathrm{T}\mathbb{R}^3$ is regular outside $\mathrm{T}_0\mathbb{R}^3$ and it is determined by two Casimirs

$$f_0(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x^2 + y^2 + z^2 \qquad f_1(x, y, z, \dot{x}, \dot{y}, \dot{z}) = x\dot{x} + y\dot{y} + z\dot{z}.$$

The tangent space $\mathrm{T}_0\mathbb{R}^3 \simeq \mathbb{R}^3$ has a linear Poisson structure

$$\dot{z}\frac{\partial}{\partial \dot{x}} \wedge \frac{\partial}{\partial \dot{y}} + \dot{x}\frac{\partial}{\partial \dot{y}} \wedge \frac{\partial}{\partial \dot{z}} + \dot{y}\frac{\partial}{\partial \dot{z}} \wedge \frac{\partial}{\partial \dot{x}}.$$

which is equal to Λ.

Example 2

In this example $M = \mathbb{R}^4$ and Λ is the following quadratic Poisson structure:

$$\Lambda = x_1 x_3 \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_2} + x_1 x_4 \frac{\partial}{\partial x_4} \wedge \frac{\partial}{\partial x_2} + x_2 x_3 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_3}$$

$$+ x_2 x_4 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial x_4} + (x_3^2 + x_4^2) \frac{\partial}{\partial x_2} \wedge \frac{\partial}{\partial x_1}.$$

This structure is degenerate at points of the linear subspace $x_3 = x_4 = 0$ and regular outside it. The symplectic foliation is the intersection of the 'book' foliation generated by vector fields $x_3 \partial/\partial x_3 + x_4 \partial/\partial x_4$, $\partial/\partial x_1$ and $\partial/\partial x_2$, consisting of three-dimensional half-spaces sewn up along the two-dimensional edge $x_3 = x_4 = 0$ of zero-dimensional leaves, and the spherical foliation.

The unit sphere $S^3 = \{x \colon \sum x_i^2 = 1\}$ we identify with the $SU(2)$ group by

$$(x_1, x_2, x_3, x_4) \longmapsto \begin{pmatrix} x_1 + \mathrm{i} x_2 & -x_3 + \mathrm{i} x_4 \\ x_3 + \mathrm{i} x_4 & x_1 - \mathrm{i} x_2 \end{pmatrix}.$$

With this identification, the tensor Λ restricted to S^3 defines a Poisson–Lie structure on $SU(2)$. This structure is quasitriangular with the generalized r-matrix

$$\mathsf{r} = \frac{\partial}{\partial x_3} \wedge \frac{\partial}{\partial x_4} \in \mathfrak{su}(2) \wedge \mathfrak{su}(2).$$

Here we used identifications

$$\mathfrak{su}(2) = T_e SU(2) \simeq T_{(1,0,0,0)} S^3$$

so in the matrix form

$$\mathsf{r} = X \wedge Y \qquad \text{with} \quad X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \in \mathfrak{su}(2).$$

It is worth noticing that this Poisson–Lie structure is related to the quantum $SU(2)$ group of Woronowicz (cf [Gr]).

Singular points on S^3 form a one-dimensional circle $x_1^2 + x_2^2 = 1$, $x_3 = x_4 = 0$ which corresponds to the Cartan subgroup (maximal torus) of $SU(2)$. At a singular point $x = (\cos \varphi, \sin \varphi, 0, 0)$, the tangent space $T_x S^3$ carries a linear Poisson structure induced by $\mathrm{d}_T \Lambda$

$$\mathrm{d}_T \Lambda|_{T_x S^3} = \cos(\varphi) \left(\dot{x}_4 \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2} + \dot{x}_3 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_2} \right)$$

$$+ \sin(\varphi) \left(\dot{x}_3 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_3} + \dot{x}_4 \frac{\partial}{\partial \dot{x}_1} \wedge \frac{\partial}{\partial \dot{x}_4} \right)$$

associated with the Lie bracket

$$\{\dot{x}_1, \dot{x}_3\} = \sin(\varphi) \dot{x}_3$$

$$\{\dot{x}_1, \dot{x}_4\} = \sin(\varphi) \dot{x}_4$$

$$\{\dot{x}_2, \dot{x}_3\} = -\cos(\varphi) \dot{x}_3$$

$$\{\dot{x}_2, \dot{x}_4\} = -\cos \varphi) \dot{x}_3.$$

In particular, on $T_e SU(2) = T_{(1,0,0,0)} S^3$ we have

$$\dot{x}_4 \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2} + \dot{x}_3 \frac{\partial}{\partial \dot{x}_3} \wedge \frac{\partial}{\partial \dot{x}_2}.$$

It follows that the cobracket λ of the tangent bialgebra of the Poisson–Lie group $SU(2)$ is given by

$$\lambda\left(\frac{\partial}{\partial \dot{x}_4}\right) = \frac{\partial}{\partial \dot{x}_4} \wedge \frac{\partial}{\partial \dot{x}_2} \qquad \lambda\left(\frac{\partial}{\partial \dot{x}_3}\right) = \frac{\partial}{\partial \dot{x}_3} \wedge \frac{\partial}{\partial \dot{x}_2} \qquad \lambda\left(\frac{\partial}{\partial \dot{x}_2}\right) = 0$$

and the associated Lie bracket on $\mathfrak{su}(2)^*$ is given by

$$\{\dot{x}_4, \dot{x}_2\} = \dot{x}_4 \qquad \{\dot{x}_3, \dot{x}_2\} = \dot{x}_3 \qquad \{\dot{x}_4, \dot{x}_3\} = 0.$$

We recognize this structure as the structure of the Lie algebra $\mathfrak{sb}(2, \mathbb{C})$ of 2×2 traceless, upper triangular complex matrices with real diagonal elements.

The introduced Poisson–Lie structure on $SU(2)$ defines then the group $SB(2, \mathbb{C})$ as the dual group. It is not difficult to verify that the corresponding double group can be identified with $SL(2, \mathbb{C})$ with $SU(2)$ and $SB(2, \mathbb{C})$ canonically embedded as subgroups.

12. Conclusions

In this paper we introduced lifts of multivector fields and related objects (like generalized foliations) from a manifold M to its tangent bundle. These operations can be considered as an extention of the tangent functor to these objects and corresponding structures. We called them tangent lifts.

Among new results are those stating that the tangent lift operator d_T on multivector fields commutes with the Schouten bracket (theorem 2.4), that the symplectic foliation of the tangent Poisson structure is the tangent foliation of the given Poisson structure (proposition 10.1), and that the tangent lift of a Poisson–Lie structure is a Poisson–Lie structure (theorem 7.1), etc. We proved also theorems describing Poisson structures by conditions for morphisms of the tangent and cotangent bundles (theorem 4.4 and theorem 4.5).

Some of results refer to already known facts, but the used methods give us new point of view, show better relations between different objects and provide deeper understandig of some well-known structures and facts as special cases of more general situations (cf. theorem 5.1 and theorem 10.2).

We are convinced that these results show the importance of the concept of tangent structures in general and of the derivation d_T in particular. The next step should be the analysis of multitangent constructions, important for classical field theory (multiphase approach) and classical mechanics, of extended objects. As we have seen, the conditions discussed in section 4 are, in fact, compatibility conditions of tangent and canonical structures. This idea can be applied in more general situations like in non-relativistic, time dependent mechanics, where the structure needed is more general than Poisson or symplectic one ([Ur]). Results of further studies on tangent lifts together with applications to analytical mechanics will be given in a separate publication.

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